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Bang–bang-type Nash equilibrium point for Markovian nonzero-sum stochastic differential game

Sur un jeu différentiel stochastique de somme non nulle avec contrôles de type bang–bang

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ABSTRACT

In this Note, we solve a nonzero-sum stochastic differential game (NZSDG) with bang–bang-type equilibrium controls by using backward stochastic differential equations (BSDEs). The generator is multi-dimensional and discontinuous with respect to z.

RÉSUMÉ

Dans cette Note, nous résolvons un jeu différentiel stochastique de somme non nulle avec contrôles d'équilibre de type bang–bang, en utilisant les équations différentielles stochastiques rétrogrades (EDSRs). Le générateur est multi-dimensionnel et discontinu par rapport à z.

Version française abrégée

Jeux différentiels stochastiques de somme non nulle de type bang–bang et EDSRs multidimensionnelles à générateurs discontinus

Cette Note traite des jeux différentiels stochastiques de somme non nulle de type bang–bang, que nous allons décrire et étudier à travers un cas spécifique, mais dont les idées sont assez générales et transposables à d'autres situations. Soit $(\Omega, \mathbf{F}, \mathbf{P})$ un espace de probabilité sur lequel est défini un mouvement brownien standard $B := (B_t)_{t \le T}$ 1-dimensionnel et qui est de filtration naturelle complétée $(\mathbf{F}_t)_{t \le T}$. Nous considérerons le cas de deux joueurs π_1 et π_2, la généralisation à plusieurs joueurs étant une simple formalité. Pour $(t, x) \in [0, T] \times \mathbf{R}$, soit $(X_s^{t,x})_{s \le T}$ le processus stochastique défini comme suit :

$$\forall s \le T, \quad X_s^{t,x} = x + (B_{s \vee t} - B_t). \tag{1}$$

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Le processus $X^{0,x}$ est la dynamique stochastique d'un système lorsqu'il n'est pas controlé. Maintenant, soit $U=[0,1]$, $V=[-1,1]$ et soit $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) l'ensemble des processus $\mathbf{F}_t$-progressivement mesurables $u=(u_t)_{t\le T}$ (resp. $v=(v_t)_{t\le T}$) sur $[0,T]\times\Omega$ et à valeurs dans U (resp. V). L'ensemble $\mathcal{M}:=\mathcal{M}_1\times\mathcal{M}_2$ est dit des contrôles admissibles pour les deux joueurs π_1 et π_2.

Soit $f:(t,x)\in[0,T]\times\mathbf{R}\to f(t,x)\in\mathbf{R}$ une fonction mesurable et à croissance linéaire, i.e. telle qu'il existe une constante C telle que $|f(t,x)|\le C(1+|x|)$, $\forall(t,x)\in[0,T]\times\mathbf{R}$, et soit Γ la fonction qui à $(t,x,u,v)\in[0,T]\times\mathbf{R}\times U\times V$ associe $\Gamma(t,x,u,v)=f(t,x)+u+v$.

À $(u,v)\in\mathcal{M}$ on associe la mesure positive $\mathbf{P}^{u,v}$ sur $(\Omega,\mathbf{F})$ définie comme suit :

$$\mathrm{d}\mathbf{P}^{u,v}=\zeta_T^{u,v}\mathrm{d}\mathbf{P}\quad\text{avec}\quad \zeta_t=1+\int_0^t\Gamma(s,X_s^{0,x},u_s,v_s)\zeta_s\,\mathrm{d}B_s,\quad t\le T.$$

Noter que $\mathbf{P}^{u,v}$ est une probabilité, puisque Γ est à croissance linéaire en x uniformément en $(u,v)\in U\times V$. Par ailleurs, il existe $p>1$ tel que $\mathbf{E}[(\zeta_T^{u,v})^p]<\infty$ ([6], p. 14). Enfin, sous $\mathbf{P}^{u,v}$ le processus $B^{u,v}=(B_s-\int_0^s\Gamma(r,X_r^{0,x},u_r,v_r)\mathrm{d}r)_{s\le T}$ est un $(\mathbf{F}_s,\mathbf{P}^{u,v})$-mouvement brownien et le processus $X^{0,x}$ vérifie :

$$\forall s\in[0,T],\quad \mathrm{d}X_s^{0,x}=\Gamma\big(s,X_s^{0,x},u_s,v_s\big)\mathrm{d}s+\mathrm{d}B_s^{u,v}\quad\text{et}\quad X_0^{0,x}=x,\tag{2}$$

i.e., $X^{0,x}$ est solution faible de l'équation différentielle stochastique (2). Lorsque le système est contrôlé avec $u=(u_s)_{s\le T}$ (resp. $v=(v_s)_{s\le T}$) par π_1 (resp. π_2), la loi de sa dynamique est la même que celle de $X^{0,x}$ sous $\mathbf{P}^{u,v}$.

Soient g_i, $i=1,2$, deux fonctions de $\mathbf{R}$ dans $\mathbf{R}$ boréliennes et à croissance polynômiale, i.e. vérifiant $|g_1(x)|+|g_2(x)|\le C(1+|x|^\gamma)$, $\forall x\in\mathbf{R}$ avec C et $\gamma>0$ constantes fixes. Lorsque le système est contrôlé par les deux joueurs avec $(u,v)\in\mathcal{M}$, on associe à π_1 (resp. π_2) un payoff égal à $J_1(u,v):=\mathbf{E}^{u,v}[g_1(X_T^{0,x})]$ (resp. $J_2(u,v):=\mathbf{E}^{u,v}[g_2(X_T^{0,x})]$), où $\mathbf{E}^{u,v}$ est l'espérance sous la probabilité $\mathbf{P}^{u,v}$.

Dans cette Note, nous nous intéressons à l'existence d'un équilibre de Nash pour ce jeu différentiel de somme non nulle, i.e. un couple de contrôle $(u^*,v^*)\in\mathcal{M}$ vérifiant $J_1(u^*,v^*)\ge J_1(u,v^*)$ et $J_2(u^*,v^*)\ge J_2(u^*,v)$, pour tout $(u,v)\in\mathcal{M}$.

Si Γ ne dépend pas de v, alors le jeu devient simplement un problème de contrôle stochastique qui admet un contrôle optimal de type bang–bang, car ce dernier ne prend ses valeurs qu'à la frontière du domaine en fonction du signe du gradient de la fonction valeur. Aussi, par analogie avec le problème de contrôle stochastique standard, dès lors que J_1 et J_2 sont sans payoffs instantanés, l'équilibre de Nash pour ce jeu, s'il existe, sera en général de type bang–bang.

Soient H_1 et H_2 les fonctions hamiltoniennes de ce jeu, i.e. les fonctions ne dépendant pas de ω, qui à $(t,x,p,q,u,v)\in[0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times U\times V$ associe :

$$H_1(t,x,p,u,v):=p\Gamma(t,x,u,v)=p\big(f(t,x)+u+v\big)\quad\text{et}\quad H_2(t,x,q,u,v):=H_1(t,x,q,u,v).\tag{3}$$

Soient maintenant $\bar{u}$ et $\bar{v}$ les fonctions définies comme suit, respectivement sur $\mathbf{R}\times U$ et $\mathbf{R}\times V$ et à valeurs dans U et V :

$$\forall p,q\in\mathbf{R},\ \epsilon_1\in U,\ \epsilon_2\in V,\quad \bar{u}(p,\epsilon_1)=\begin{cases}1 & \text{si } p>0,\\ 0 & \text{si } p<0,\\ \epsilon_1 & \text{si } p=0,\end{cases}\quad\text{et}\quad \bar{v}(q,\epsilon_2)=\begin{cases}1 & \text{si } q>0,\\ -1 & \text{si } q<0,\\ \epsilon_2 & \text{si } q=0.\end{cases}\tag{4}$$

Alors $\bar{u}$ et $\bar{v}$ vérifient la condition suivante, dite de Isaacs généralisée : pour tout $(t,x,p,q,u,v)\in[0,T]\times\mathbf{R}\times\mathbf{R}\times U\times V$ et $(\epsilon_1,\epsilon_2)\in U\times V$,

$$\begin{aligned}&H_1^*(t,x,p,q,\epsilon_2):=H_1\big(t,x,p,\bar{u}(p,\epsilon_1),\bar{v}(q,\epsilon_2)\big)\ge H_1\big(t,x,p,u,\bar{v}(q,\epsilon_2)\big)\quad\text{et}\\ &H_2^*(t,x,p,q,\epsilon_1):=H_2\big(t,x,q,\bar{u}(p,\epsilon_1),\bar{v}(q,\epsilon_2)\big)\ge H_2\big(t,x,q,\bar{u}(p,\epsilon_1),v\big).\end{aligned}\tag{5}$$

On notera que la fonction H_1^* (resp. H_2^*) ne dépend pas de ϵ_1 (resp. ϵ_2), puisque $p\bar{u}(p,\epsilon_1)$ (resp. $q\bar{v}(q,\epsilon_2)$) ne dépend pas de ϵ_1 (resp. ϵ_2).

Pour étudier ce problème de jeu différentiel stochastique de somme non nulle, nous allons utiliser les équations différentielles stochastiques rétrogrades comme dans plusieurs travaux antécédants sur ce sujet [4,5]. Le lien est donné par les deux résultats suivants :

Proposition 1. *Pour tout $(u,v)\in\mathcal{M}$, pour $i=1,2$, il existe un couple de processus $(Y^{i,(u,v)},Z^{i,(u,v)})$, $\mathbf{F}_t$-progressivement mesurables et à valeurs dans $\mathbf{R}\times\mathbf{R}$, vérifiant :*

(i) *pour tout $q\ge 1$,*

$$\mathbf{E}^{u,v}\left[\sup_{0\le s\le T}\big|Y_s^{i,(u,v)}\big|^q+\left(\int_0^T\big|Z_s^{i,(u,v)}\big|^2\mathrm{d}s\right)^{\frac{q}{2}}\right]<\infty.\tag{6}$$

(ii) $\forall t \le T$,

$$Y_t^{i,(u,v)} = g_i\big(X_T^{0,x}\big) + \int_t^T H_i\big(s, X_s^{0,x}, Z_s^{i,(u,v)}, u_s, v_s\big)\mathrm{d}s - \int_t^T Z_s^{i,(u,v)}\mathrm{d}B_s. \tag{7}$$

(iii) $Y_0^{i,(u,v)} = J_i(u,v)$.

Proposition 2. *Supposons qu'il existe u^1, u^2, (Y^1, Z^1), (Y^2, Z^2) et θ^1, θ^2 tels que :*

(i) *u^1 et u^2 sont deux fonctions déterministes, mesurables et à croissance polynômiale de $[0,T] \times \mathbf{R}$ dans $\mathbf{R}$,*
(ii) *(Y^1, Z^1) et (Y^2, Z^2) sont deux couples de processus $\mathbf{F}_t$-progressivement mesurables à valeurs dans $\mathbf{R}^{1+1}$,*
(iii) *θ^1 (resp. θ^2) est un processus $\mathbf{F}_t$-progressivement mesurable à valeurs dans U (resp. V),*

et vérifiant :

(a) *$\mathbf{P}$-p.s., $\forall s \le T$, $Y_s^i = u^i(s, X_s^{0,x})$ et $Z^i(\omega) := (Z_s^i(\omega))_{s\le T}$ est $\mathrm{d}s$-carré intégrable ;*
(b) *Pour tout $s \le T$,*

$$\begin{cases} -\mathrm{d}Y_s^1 = H_1^*\big(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s^2\big)\,\mathrm{d}s - Z_s^1\mathrm{d}B_s, & Y_T^1 = g_1\big(X_T^{0,x}\big)\,; \\ -\mathrm{d}Y_s^2 = H_2^*\big(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s^1\big)\,\mathrm{d}s - Z_s^2\mathrm{d}B_s, & Y_T^2 = g_2\big(X_T^{0,x}\big). \end{cases} \tag{8}$$

Alors la paire de contrôle $(\bar{u}(Z_s^1, \theta_s^1), \bar{v}(Z_s^2, \theta_s^2))_{s\le T}$ est un point d'équilibre de Nash de type bang–bang du jeu différentiel stochastique de somme non nulle.

La preuve de ce dernier résultat est assez aisée à obtenir par comparaison des solutions d'EDSR après avoir fait un changement de probabilité et en utilisant le fait que $(\bar{u}, \bar{v})$ vérifient (5). Cependant, la difficulté essentielle réside dans la preuve de l'existence d'une solution à l'EDSR (8), qui est de type multidimensionnelle et à générateur discontinu en (z_1, z_2). La discontinuité de H_1^* (resp. H_2^*) en (p,q) provient de la discontinuité de $\bar{v}$ (resp. $\bar{u}$) en $q = 0$ (resp. $p = 0$). Dans la suite, nous allons montrer que (8) a une solution qui produira donc un point d'équilibre pour le jeu.

Existence of a Nash equilibrium for the bang–bang-type NZSDG

Similar nonzero-sum differential game of unsmooth type as we presented in the French Abridged Version has been studied by G.J. Olsder [8] in the deterministic case. Recent works on this subject, also in the deterministic case, include papers by P. Cardaliaguet and S. Plaskacz [2], and by P. Cardaliaguet [1], which show that there exists a unique Nash equilibrium payoff of feedback form. But this equilibrium payoff depends in a very unstable way on the terminal data. Besides, it is not obvious to generalize the result in [2] to higher dimensions. The stochastic case has been analyzed by P. Mannucci [7] with the help of a system of Hamilton–Jacobi equations and related parabolic PDE techniques. Notice that the state process in [7] belongs to a bounded domain. However, some techniques of PDE in the global domain are not so straightforward.

The main novelty of this note is that we show the existence of a Nash equilibrium point of bang–bang type to a nonzero-sum stochastic differential game in a global domain. Moreover, the result and the techniques in this note can be generalized to the multiple dimensions directly. However, the existence of NEP of feedback form is still an open problem.

As pointed out previously, it remains to show the existence of a solution for the multidimensional BSDE (8) with the appropriate properties. This is the main aim of this section.

Theorem 1. *There exist u^1, u^2, (Y^1, Z^1), (Y^2, Z^2) and θ^1, θ^2 that satisfy (i)–(iii) and (a), (b) of Proposition 2.*

Proof. Step 1. First let us point out that the functions $p \in \mathbf{R} \mapsto p\bar{u}(p, \epsilon_1)$ and $q \in \mathbf{R} \mapsto q\bar{v}(q, \epsilon_2)$ are Lipschitz uniformly with respect to ϵ_1 and ϵ_2, since $p\bar{u}(p,\epsilon_1) = p\bar{u}(p,0) = \sup_{u\in U} pu$ and $q\bar{v}(q,\epsilon_2) = q\bar{v}(q,0) = \sup_{v\in V} qv$. Hereafter $\bar{u}(p,0)$ (resp. $\bar{v}(q,0)$) will be simply denoted by $\bar{u}(p)$ (resp. $\bar{v}(p)$). Next for $n \ge 1$, let $\bar{u}^n$ and $\bar{v}^n$ be the functions defined as follows:

$$\bar{u}^n(p) = \begin{cases} 0 & \text{if } p \le -1/n, \\ 1 & \text{if } p \ge 0, \\ np+1 & \text{if } p \in (-1/n, 0), \end{cases} \quad \text{and} \quad \bar{v}^n(q) = \begin{cases} -1 & \text{if } q \le -1/n, \\ 1 & \text{if } q \ge 1/n, \\ nq & \text{if } q \in (-1/n, 1/n). \end{cases}$$

Finally let Φ_n be the function $x \in \mathbf{R} \mapsto \Phi_n(x) = (x \wedge n) \vee (-n) \in \mathbf{R}$. Note that Φ_n is bounded by n while $\bar{u}^n$ and $\bar{v}^n$ are Lipschitz in p and q respectively and roughly speaking they are approximations of $\bar{u}$ and $\bar{v}$.

Now for $n \geq 1$ let $(Y^{i,n;t,x}, Z^{i,n;t,x})$, $i = 1, 2$, be the solution in $\mathcal{S}_T^2(\mathbf{R}) \times \mathcal{H}_T^2(\mathbf{R})$[1] of the following BSDE of dimension two. The solution exists thanks to the result by Pardoux and Peng [9] since the generator of the BSDE is Lipschitz in (p, q). For any $s \in [t, T]$,

$$\begin{cases} Y_s^{1,n;t,x} = g_1\big(X_T^{t,x}\big) + \int_s^T \big\{\Phi_n\big(Z_r^{1,n;t,x}\big)\Phi_n\big(f\big(r, X_r^{t,x}\big)\big) + \Phi_n\big(Z_r^{1,n;t,x}\bar{u}\big(Z_r^{1,n;t,x}\big)\big) + \Phi_n\big(Z_r^{1,n;t,x}\big)\bar{v}^n\big(Z_r^{2,n;t,x}\big)\big\}\mathrm{d}r \\ \qquad - \int_s^T Z_r^{1,n;t,x}\mathrm{d}B_r; \\ Y_s^{2,n;t,x} = g_2\big(X_T^{t,x}\big) + \int_s^T \big\{\Phi_n\big(Z_r^{2,n;t,x}\big)\Phi_n\big(f\big(r, X_r^{t,x}\big)\big) + \Phi_n\big(Z_r^{2,n;t,x}\bar{v}\big(Z_r^{2,n;t,x}\big)\big) + \Phi_n\big(Z_r^{2,n;t,x}\big)\bar{u}^n\big(Z_r^{1,n;t,x}\big)\big\}\mathrm{d}r \\ \qquad - \int_s^T Z_r^{2,n;t,x}\mathrm{d}B_r. \end{cases} \tag{9}$$

Using the result by El-Karoui et al. ([3], p. 46, Theorem 4.1), there exist measurable deterministic functions $u^{i,n}$ and $d^{i,n}$ of $(t, x) \in [0, T] \times \mathbf{R}$, $i = 1, 2$ and $n \geq 1$, such that:

$$\forall s \in [t, T], \quad Y_s^{i,n;t,x} = u^{i,n}\big(s, X_s^{t,x}\big) \quad \text{and} \quad Z_s^{i,n;t,x} = d^{i,n}\big(s, X_s^{t,x}\big). \tag{10}$$

Moreover, for $n \geq 1$ and $i = 1, 2$, the functions $u^{i,n}$ verify: $\forall (t, x) \in [0, T] \times \mathbf{R}$,

$$u^{i,n}(t, x) = \mathbf{E}\left[g_i\big(X_T^{t,x}\big) + \int_t^T H_i^n\big(r, X_r^{t,x}\big)\mathrm{d}r\right] \tag{11}$$

with, for any $(s, x) \in [0, T] \times \mathbf{R}$,

$$\begin{cases} H_1^n(s, x) = \Phi_n\big(d^{1,n}(s, x)\big)\Phi_n\big(f(s, x)\big) + \Phi_n\big(d^{1,n}(s, x)\bar{u}\big(d^{1,n}(s, x)\big)\big) + \Phi_n\big(d^{1,n}(s, x)\big)\bar{v}^n\big(d^{2,n}(s, x)\big); \\ H_2^n(s, x) = \Phi_n\big(d^{2,n}(s, x)\big)\Phi_n\big(f(s, x)\big) + \Phi_n\big(d^{2,n}(s, x)\bar{v}\big(d^{2,n}(s, x)\big)\big) + \Phi_n\big(d^{2,n}(s, x)\big)\bar{u}^n\big(d^{1,n}(s, x)\big). \end{cases} \tag{12}$$

Step 2. We provide, in this step, the uniform bound of processes $(Y^{i,n;t,x}, Z^{i,n;t,x})$ with respect to n for $i = 1, 2$, $(t, x) \in [0, T] \times \mathbf{R}$. First, note that for $i = 1, 2$, $H_i^n(s, X_s^{t,x})$ are the generators of BSDE (9) and $|H_i^n(s, X_s^{t,x})| \leq \Phi_n(C(1 + |X_s^{t,x}|))|Z_s^{i,n;t,x}| + C|Z_s^{i,n;t,x}|$ for $n \geq 1$, $(t, x) \in [0, T] \times \mathbf{R}$. Let us now consider the following BSDEs, for $i = 1, 2$ and $s \in [t, T]$,

$$\bar{Y}_s^{i,n} = g_i\big(X_T^{t,x}\big) + \int_s^T \big\{\Phi_n\big(C\big(1 + \big|X_r^{t,x}\big|\big)\big)\big|\bar{Z}_r^{i,n}\big| + C\big|\bar{Z}_r^{i,n}\big|\big\}\mathrm{d}r - \int_s^T \bar{Z}_r^{i,n}\mathrm{d}B_r. \tag{13}$$

Observing that the application $z \in \mathbf{R} \mapsto \Phi_n(C(1 + |X_r^{t,x}|))|z| + C|z|$ is Lipschitz continuous, therefore the solution $(\bar{Y}^{i,n}, \bar{Z}^{i,n})$ of the above BSDE exists on space $\mathcal{S}_T^2(\mathbf{R}) \times \mathcal{H}_T^2(\mathbf{R})$ and is unique. If we show the uniform estimate for $\bar{Y}^{i,n}$ w.r.t. n, then the estimate for $Y^{i,n;t,x}$ will be a straightforward consequence of the comparison theorem of BSDEs. Below, we will focus on the property of $\bar{Y}^{i,n}$. Using again the result by El-Karoui et al. ([3], p. 46, Theorem 4.1) yields that there exist deterministic measurable functions $\bar{u}^{i,n} : [0, T] \times \mathbf{R} \to \mathbf{R}$ such that, for any $s \in [t, T]$,

$$\bar{Y}_s^{i,n} = \bar{u}^{i,n}\big(s, X_s^{t,x}\big), \quad i = 1, 2. \tag{14}$$

Next let us consider the process $B_s^{i,n} = B_s - \int_0^s [\Phi_n(C(1 + |X_r^{t,x}|)) + C]\,\mathrm{sign}(Z_r^{i,n})\mathrm{d}r$, $t \leq s \leq T$, $i = 1, 2$, which is, thanks to Girsanov's Theorem, a Brownian motion under the probability $\mathbf{P}^{i,n}$ on $(\Omega, \mathcal{F})$ whose density with respect to $\mathbf{P}$ is $\zeta_T\{[\Phi_n(C(1 + |X_s^{t,x}|)) + C]\,\mathrm{sign}(Z_s^{i,n})\}$ where for any $z \in \mathbf{R}$, $\mathrm{sign}(z) = 1_{\{|z| \neq 0\}}\frac{z}{|z|}$ and $\zeta_t(.)$ verifies $\zeta_t(\Theta) = 1 + \int_0^t \Theta_s\zeta_s\,\mathrm{d}B_s$, $t \leq T$. Then the BSDE (13) will be simplified into

$$\bar{Y}_s^{i,n} = g_i\big(X_T^{t,x}\big) - \int_s^T \bar{Z}_r^{i,n}\mathrm{d}B_r^{i,n}, \quad s \in [t, T],\ i = 1, 2.$$

[1] The space $\mathcal{H}_T^p(\mathbf{R})$ $(p \geq 1)$ is of $\mathbf{R}$-valued processes $(Z_s)_{0 \leq s \leq T}$ that satisfy $\mathbf{E}[(\int_0^T |Z_s|^2\mathrm{d}s)^{p/2}] < \infty$; $\mathcal{S}_T^2(\mathbf{R})$ is the space of $\mathbf{R}$-valued process $(\zeta_s)_{0 \leq s \leq T}$, $\mathbf{F}$-adapted, continuous and satisfying $\mathbf{E}[\sup_{s \leq T}|\zeta_s|^2] < \infty$.

In view of (14), we obtain $\bar{u}^{i,n}(t,x) = \mathbf{E}^{i,n}[g_i(X_T^{t,x})|\mathbf{F}_t]$, $i=1,2$, where $\mathbf{E}^{i,n}$ is the expectation under probability $\mathbf{P}^{i,n}$. By taking the expectation on both sides under the probability $\mathbf{P}^{i,n}$ and considering $\bar{u}^{i,n}(t,x)$ is deterministic, we arrive at

$$\bar{u}^{i,n}(t,x) = \mathbf{E}^{i,n}\big[\mathbf{E}^{i,n}\big[g_i\big(X_T^{t,x}\big)|\mathbf{F}_t\big]\big] = \mathbf{E}^{i,n}\big[g_i\big(X_T^{t,x}\big)\big], \quad i=1,2.$$

The functions g_i, $i=1,2$, are of polynomial growth, combining with the result of U.G. Haussmann ([6], p. 14) gives that there exists a constant $p_0 \in (1,2)$ (which does not depend on (t,x)) such that:

$$\begin{aligned}
\big|\bar{u}^{i,n}(t,x)\big| &\le C\mathbf{E}^{i,n}\Big[\sup_{s\in[t,T]}\big\{1+\big|X_s^{t,x}\big|^\gamma\big\}\Big] \\
&= C\mathbf{E}\Big[\Big(\sup_{s\in[t,T]}\big\{1+\big|X_s^{t,x}\big|^\gamma\big\}\Big)\big(\zeta_T\big\{\big[\Phi_n\big(C\big(1+\big|X_s^{t,x}\big|\big)\big)+C\big]\operatorname{sign}\big(Z_s^{i,n}\big)\big\}\big)\Big] \\
&\le C\mathbf{E}\Big[\sup_{s\in[t,T]}\big(1+\big|X_s^{t,x}\big|^\gamma\big)^{\frac{p_0}{p_0-1}}\Big] + C\mathbf{E}\big[\big(\zeta_T\big\{\big[\Phi_n\big(C\big(1+\big|X_s^{t,x}\big|\big)\big)+C\big]\operatorname{sign}\big(Z_s^{i,n}\big)\big\}\big)^{p_0}\big] \\
&\le C\big(1+|x|^\lambda\big),
\end{aligned}$$

with the constant $\lambda \ge 1$. It follows from comparison of solutions of BSDEs, for any $s \in [t,T]$ and $i=1,2$, $\bar{Y}_s^{i,n} = \bar{u}^{i,n}(s, X_s^{t,x}) \ge Y_s^{i,n} = u^{i,n}(s, X_s^{t,x})$. Choosing $s=t$ leads to $u^{i,n}(t,x) \le C(1+|x|^\lambda)$, $(t,x)\in[0,T]\times\mathbf{R}$. In a similar way, we can show that $u^{i,n}(t,x) \ge -C(1+|x|^\lambda)$, $(t,x)\in[0,T]\times\mathbf{R}$. Therefore, $u^{i,n}$, $i=1,2$ are of polynomial growth with respect to (t,x) uniformly in n, i.e. $|u^{i,n}(t,x)| \le C(1+|x|^\lambda)$, $i=1,2$, $(t,x)\in[0,T]\times\mathbf{R}$. Then, it follows from (10) and the polynomial growth of function $u^{i,n}$, $i=1,2$, that there exists a constant C that does not depend on n, such that, for any $\alpha \ge 1$,

$$\mathbf{E}\Big[\sup_{t\le s\le T}\big|Y_s^{i,n;t,x}\big|^\alpha\Big] \le C. \tag{15}$$

Next using Itô's formula with $(Y_s^{i,n;t,x})^2$ yields the existence of a constant C which is not depending on n such that,

$$\mathbf{E}\Bigg[\int_t^T \big|Z_r^{i,n;t,x}\big|^2\,\mathrm{d}r\Bigg] \le C. \tag{16}$$

Step 3. We show the convergence result in this step. Observing (12) and (16), we infer that the processes $H_i^n(s, X_s^{0,x})$, $i=1,2$ are uniformly bounded in $L^q(\mathrm{d}t\otimes\mathrm{d}\mathbf{P})$ for any constant $q\in(1,2)$ and fixed $x\in\mathbf{R}$, i.e.,

$$\mathbf{E}\Bigg[\int_0^T \big|H_i^n\big(s, X_s^{0,x}\big)\big|^q\,\mathrm{d}s\Bigg] = \int_{[0,T]\times\mathbf{R}} \big|H_i^n(s,y)\big|^q \mu(0,x;s,\mathrm{d}y)\,\mathrm{d}s \le C, \tag{17}$$

where $\mu(0,x;s,\mathrm{d}x)$ is the law of $X_s^{0,x}$. As a result, there exists a sub-sequence $\{n_k\}$ (we still denote it by $\{n\}$ for simplicity) and a $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbf{R})$-measurable deterministic function $H_i(s,x)$ such that:

$$H_i^n \to H_i \text{ weakly in } L^q\big([0,T]\times\mathbf{R};\,\mu(0,x;s,\mathrm{d}x)\,\mathrm{d}s\big), \quad \text{for } i=1,2,\ q\in(1,2). \tag{18}$$

Let us now illustrate how we pass from the weak convergence to strong convergence. Take $i=1$ for example. For each $(t,x)\in[0,T]\times\mathbf{R}$, $\delta>0$ and integer k,m,n, using (11) yields,

$$\begin{aligned}
\big|u^{1,n}(t,x) - u^{1,m}(t,x)\big| \le{}& \Bigg|\mathbf{E}\Bigg[\int_t^{t+\delta} H_1^n\big(s,X_s^{t,x}\big) - H_1^m\big(s,X_s^{t,x}\big)\,\mathrm{d}s\Bigg]\Bigg| \\
&+ \Bigg|\mathbf{E}\Bigg[\int_{t+\delta}^T \big(H_1^n\big(s,X_s^{t,x}\big) - H_1^m\big(s,X_s^{t,x}\big)\big)1_{\{|X_s^{t,x}|\le k\}}\mathrm{d}s\Bigg]\Bigg| \\
&+ \Bigg|\mathbf{E}\Bigg[\int_{t+\delta}^T \big(H_1^n\big(s,X_s^{t,x}\big) - H_1^m\big(s,X_s^{t,x}\big)\big)1_{\{|X_s^{t,x}|> k\}}\mathrm{d}s\Bigg]\Bigg|.
\end{aligned}$$

The first term on the right-hand side is bounded by $C\delta^{\frac{q-1}{q}}$, which is obtained by Young's inequality and (17). For the third component, it follows from Young's inequality, Markov inequality and the result (17) that it is bounded by $Ck^{-\frac{q-1}{q}}$, while the second part is exactly the following one,

$$\left|\int_{\mathbf{R}}\int_{t+\delta}^{T}\big(H_1^n(s,\eta)-H_1^m(s,\eta)\big)1_{\{|\eta|\le k\}}\frac{1}{\sqrt{2\pi(s-t)}}\,\mathrm{e}^{-\frac{(\eta-x)^2}{2(s-t)}}\,\mathrm{d}s\,\mathrm{d}\eta\right|$$
$$=\left|\int_{\mathbf{R}}\int_{t+\delta}^{T}\big(H_1^n(s,\eta)-H_1^m(s,\eta)\big)1_{\{|\eta|\le k\}}\frac{\sqrt{s}}{\sqrt{s-t}}\,\mathrm{e}^{-\frac{(\eta-x)^2t}{2s(s-t)}}\frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{(\eta-x)^2}{2s}}\,\mathrm{d}s\,\mathrm{d}\eta\right|.$$

But for any (t,x) the function $(s,\eta)\in[t+\delta,T]\times\mathbf{R}\mapsto\frac{\sqrt{s}}{\sqrt{s-t}}e^{-\frac{(\eta-x)^2t}{2s(s-t)}}$ is bounded. Thus by the weak convergence result (18), we have that the term above converges to 0 as $n,m\to\infty$. Therefore, $(u^{i,n}(t,x))_{n\ge1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}$. Then there exists a measurable application u^1 on $[0,T]\times\mathbf{R}$ such that $\lim_{n\to\infty}u^{1,n}(t,x)=u^1(t,x)$ for each $(t,x)\in[0,T]\times\mathbf{R}$.

Taking into account (10) and the polynomial growth property of function $u^{1,n}$, we obtain by Lebesgue dominated convergence theorem that $((Y_s^{1,n;0,x}=u^{1,n}(s,X_s^{0,x}))_{s\le T})_{n\ge1}$ converges to $(Y_s^1=u^1(s,X_s^{0,x}))_{s\le T}$ in $\mathcal{H}_T^2(\mathbf{R})$. Moreover, using Itô's formula with $(Y_s^{1,n;0,x}-Y_s^{1,m;0,x})^2_{s\le T}$, we get in a standard way that $((Z_s^{1,n;0,x}=d^{1,n}(s,X_s^{0,x}))_{s\le T})_{n\ge1}$ is convergent to a process $(Z_s^1)_{s\le T}$ in $\mathcal{H}_T^2(\mathbf{R})$. Then, we extract a subsequence still denoted by $(Z^{1,n})_{n\ge1}$ that converges to Z^1, $\mathrm{d}t\otimes\mathrm{d}\mathbf{P}$-a.e. and such that $\sup_{n\ge1}|Z_t^{1,n}(\omega)|\in\mathcal{H}_T^2(\mathbf{R})$. Repeating the previous procedure for processes $(Y_s^{2,n;0,x},Z_s^{2,n;0,x})_{s\le T}$ of player π_2 gives the existence of processes $(Y_s^2,Z_s^2)_{s\le T}$. Finally, we have also the convergence in $\mathcal{S}_T^2(\mathbf{R})$ of the sequence $(Y^{1,n;0,x})_{n\ge0}$ (resp. $(Y^{2,n;0,x})_{n\ge0}$) toward Y^1 (resp. Y^2) and thus Y^1 and Y^2 are continuous processes.

Step 4. Finally, let us show the existence of processes θ^2 (resp. θ^1) valued on V (resp. U) such that Y^i,Z^i,u^i,θ^i, $i=1,2$, verify the requirements of Proposition 2. In this step, we delete the superscript $0,x$ for convenience.

First recall H_1^n of (12) and note that $\Phi_n(Z_s^{1,n})\Phi_n(f(s,X_s))+\Phi_n(Z_s^{1,n}\bar{u}(Z_s^{1,n}))\to_{n\to\infty}Z_s^1f(s,X_s)+Z_s^1\bar{u}(Z_s^1)\,\mathrm{d}s\otimes\mathrm{d}\mathbf{P}$-a.e. since $Z^{1,n}\to_{n\to\infty}Z^1$, $\mathrm{d}s\otimes\mathrm{d}\mathbf{P}$, for any $x\in\mathbf{R}$, $\Phi_n(x)\to_{n\to\infty}x$ and finally by the continuity of $p\in\mathbf{R}\mapsto p\bar{u}(p)$. The rest part in the expression of H_1^n is:

$$\Phi_n\big(Z_s^{1,n}\big)\bar{v}^n\big(Z_s^{2,n}\big)=\Phi_n\big(Z_s^{1,n}\big)\bar{v}^n\big(Z_s^{2,n}\big)1_{\{Z_s^2\neq0\}}+\Phi_n\big(Z_s^{1,n}\big)\bar{v}^n\big(Z_s^{2,n}\big)1_{\{Z_s^2=0\}}.$$

The term $\Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2\neq0\}}$ converges to $Z_s^1\bar{v}(Z_s^2)1_{\{Z_s^2\neq0\}}\mathrm{d}s\otimes\mathrm{d}\mathbf{P}$-a.e., since $\bar{v}$ is continuous at any point different from 0 and $Z_s^{2,n}\to Z_s^2\mathrm{d}s\otimes\mathrm{d}\mathbf{P}$-a.e. Let us next define an $\mathbf{F}_t$-progressively measurable process $(\theta_s^2)_{s\le T}$ valued on V as the weak limit in $\mathcal{H}_T^2(\mathbf{R})$ of some subsequence $((\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}})_{s\le T})_{k\ge0}$. The weak limit exists since $\bar{v}^{n_k}$ is bounded. Then the sequence $((\Phi_n(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}})_{s\le T})_{k\ge0}$ converges weakly in $\mathcal{H}_T^2(\mathbf{R})$ to $(Z_s^1\theta_s^21_{\{Z_s^2=0\}})_{s\le T}$. Therefore $(H_1^{n_k}(s,X_s))_{k\ge0}$ converges weakly to $H_1^*(s,X_s,Z_s^1,Z_s^2,\theta_s^2)$ in $\mathcal{H}_T^2(\mathbf{R})$.

Let us now show for any stopping time τ, we have:

$$\int_0^\tau H_1^{n_k}(s,X_s)\,\mathrm{d}s\to_{k\to\infty}\int_0^\tau H_1^*\big(s,X_s,Z_s^1,Z_s^2,\theta_s^2\big)\,\mathrm{d}s\quad\text{weakly in }L^2(\Omega,\mathrm{d}\mathbf{P}).\tag{19}$$

As explained at the beginning of this step, we only need to show the weak convergence of $\int_0^\tau\Phi_{n_k}(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}\mathrm{d}s$ to $\int_0^\tau Z_s^1\theta_s^21_{\{Z_s^2=0\}}\mathrm{d}s$ in $L^2(\Omega,d\mathbf{P})$ as $k\to\infty$. Obviously, we have:

$$\int_0^\tau\Phi_{n_k}\big(Z_s^{1,n_k}\big)\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s=\int_0^\tau\big(\Phi_{n_k}\big(Z_s^{1,n_k}\big)-Z_s^1\big)\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s+\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s.$$

On the right-hand side, the first term converges to 0 by Lebesgue's dominated convergence theorem. Below, we will give the weak convergence in $L^2(\Omega,\mathrm{d}\mathbf{P})$ of the second term. For any random variable $\xi\in L^2(\Omega,\mathbf{F}_T,\mathrm{d}\mathbf{P})$, we need to show:

$$\mathbf{E}\left[\xi\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s\right]\to\mathbf{E}\left[\xi\int_0^\tau Z_s^1\theta_s^21_{\{Z_s^2=0\}}\mathrm{d}s\right]\quad\text{as }k\to\infty.$$

Thanks to martingale representation theorem, there exists an $\mathbf{F}_T$-progressively measurable process $(\eta_s)_{s\le T}\in\mathcal{H}_T^2(\mathbf{R})$ such that, $\xi=\mathbf{E}[\xi]+\int_0^T\eta_s\mathrm{d}B_s$. Therefore,

$$\mathbf{E}\bigg[\xi\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]=\mathbf{E}\bigg[\mathbf{E}[\xi]\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]+\mathbf{E}\bigg[\int_0^\tau \eta_s\mathrm{d}B_s\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s\bigg].$$

Notice that $\mathbf{E}[\xi]\mathbf{E}[\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}\mathrm{d}s]$ converges to $\mathbf{E}[\xi]\mathbf{E}[\int_0^\tau Z_s^1\theta_s^2 1_{\{Z_s^2=0\}}\mathrm{d}s]$ as $k\to\infty$, since $(Z_s^1)_{s\le T}\in\mathcal{H}_T^2(\mathbf{R})$ and $\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}\to_{k\to\infty}\theta_s^2$ weakly in $\mathcal{H}_T^2(\mathbf{R})$. Next, by Itô's formula,

$$\mathbf{E}\bigg[\int_0^\tau \eta_s\,\mathrm{d}B_s\cdot\int_0^\tau Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\,\mathrm{d}s\bigg]=\mathbf{E}\bigg[\int_0^\tau\bigg(\int_0^s\eta_u\mathrm{d}B_u\bigg)Z_s^1\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]+\mathbf{E}\bigg[\int_0^\tau\bigg(\int_0^s Z_u^1\bar{v}^{n_k}\big(Z_u^{2,n_k}\big)1_{\{Z_u^2=0\}}\mathrm{d}u\bigg)\eta_s\,\mathrm{d}B_s\bigg].$$

The later one on the right-hand side is 0, since $\int_0^\cdot(\int_0^s Z_u^1\bar{v}^{n_k}(Z_u^{2,n_k})1_{\{Z_u^2=0\}}\mathrm{d}u)\eta_s\mathrm{d}B_s$ is an $\mathbf{F}_t$-martingale. For the former part, denote $\int_0^s\eta_u\mathrm{d}B_u$ by ψ_s for any $s\in[0,\tau]$. Then for any integer $\kappa>0$, we have:

$$\bigg|\mathbf{E}\bigg[\int_0^\tau\psi_s Z_s^1\big(\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)-\theta_s^2\big)1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]\bigg|\le\bigg|\mathbf{E}\bigg[\int_0^\tau\psi_s Z_s^1\big(\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)-\theta_s^2\big)1_{\{|\psi_s Z_s^1|\le\kappa\}}\cdot 1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]\bigg|+\bigg|\mathbf{E}\bigg[\int_0^\tau\psi_s Z_s^1\big(\bar{v}^{n_k}\big(Z_s^{2,n_k}\big)-\theta_s^2\big)1_{\{|\psi_s Z_s^1|\ge\kappa\}}\cdot 1_{\{Z_s^2=0\}}\mathrm{d}s\bigg]\bigg|.$$

On the right-hand side of the above equation, the first component converges to 0, which is the consequence of $\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}\to_{k\to\infty}\theta_s^2$ weakly in $\mathcal{H}_T^2(\mathbf{R})$. For the second component, considering both $(\bar{v}^{n_k}(Z_s^{2,n_k}))_{s\le\tau}$ and $(\theta_s^2)_{s\le\tau}$ are bounded, it is smaller than $C|\mathbf{E}[\int_0^\tau|\psi_s Z_s^1|1_{\{|\psi_s Z_s^1|\ge\kappa\}}\mathrm{d}s]|$, which obviously converges to 0 as $\kappa\to\infty$. Thus (19) holds.

Besides, we also have $\int_0^\tau Z_s^{1,n_k}\mathrm{d}B_s\to_{k\to\infty}\int_0^\tau Z_s^1\mathrm{d}B_s$ in $L^2(\Omega,\mathrm{d}\mathbf{P})$, which is obtained from the convergence of $(Z^{1,n_k})_{k\ge0}$ to Z^1 in $\mathcal{H}_T^2(\mathbf{R})$ and the isometric property. Then by observing the approximation BSDE (9) in a forward way, i.e. for any stopping time τ,

$$Y_\tau^{1,n_k}=Y_0^{1,n_k}-\int_0^\tau H_1^{n_k}(s,X_s)\,\mathrm{d}s+\int_0^\tau Z_s^{1,n_k}\mathrm{d}B_s,$$

combining with the convergence of $((Y_s^{1,n_k})_{s\le T})_{k\ge0}$ to $(Y_s^1)_{s\le T}$ in $\mathcal{S}_T^2(\mathbf{R})$, we infer that

$$\mathbf{P}\text{-a.s.}\quad Y_\tau^1=Y_0^1-\int_0^\tau H_1^*\big(s,X_s,Z_s^1,Z_s^2,\theta_s^2\big)\,\mathrm{d}s+\int_0^\tau Z_s^1\,\mathrm{d}B_s\quad\text{for every stopping time }\tau.$$

As τ is arbitrary, then the processes appearing in the two sides are indistinguishable, i.e., $\mathbf{P}$-a.s.

$$\forall t\le T,\quad Y_t^1=Y_0^1-\int_0^t H_1^*\big(s,X_s,Z_s^1,Z_s^2,\theta_s^2\big)\,\mathrm{d}s+\int_0^t Z_s^1\,\mathrm{d}B_s.$$

On the other hand, $Y_T^1=g_1(X_T)$, then

$$\mathbf{P}\text{-a.s.}\quad\forall t\le T,\quad Y_t^1=g_1(X_T)+\int_t^T H_1^*\big(s,X_s,Z_s^1,Z_s^2,\theta_s^2\big)\mathrm{d}s-\int_t^T Z_s^1\,\mathrm{d}B_s.$$

Similarly, for player π_2, there exists an $\mathbf{F}_t$-progressively measurable process $(\theta_s^1)_{s\le T}$ valued on U, such that

$$\mathbf{P}\text{-a.s.}\quad \forall t \le T,\quad Y_t^2 = g_2(X_T) + \int_t^T H_2^*\big(s, X_s, Z_s^1, Z_s^2, \theta_s^1\big)\,\mathrm{d}s - \int_t^T Z_s^2\,\mathrm{d}B_s.$$

The proof is completed. □

As a result of Theorem 1 and Proposition 2, we obtain the main result of this Note.

Theorem 2. *The bang–bang nonzero-sum stochastic differential game has a Nash equilibrium point.*

Remark 1. Theorem 2 can be generalized to the following frameworks:

(i) in the drift term Γ, one can replace u (resp. v) by $h(u)$ (resp. $l(v)$) where h and l are continuous functions;
(ii) the dynamics of the process $X^{0,x}$ of (1) may contain a diffusion term σ with appropriate properties;
(iii) in the same way one can deal with multidimensional diffusion processes $X^{0,x}$.

References

[1] P. Cardaliaguet, On the instability of the feedback equilibrium payoff in a nonzero-sum differential game on the line, in: Advances in Dynamic Game Theory, Birkhäuser, Boston, MA, USA, 2007, pp. 57–67.
[2] P. Cardaliaguet, P. Slawomir, Existence and uniqueness of a Nash equilibrium feedback for a simple nonzero-sum differential game, Int. J. Game Theory 32 (1) (2003) 33–71.
[3] N. El-Karoui, S. Peng, M.-C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1) (1997) 1–71.
[4] S. Hamadène, Nonzero sum linear-quadratic stochastic differential games and backward–forward equations, Stoch. Anal. Appl. 17 (1) (1999) 117–130.
[5] S. Hamadène, J.-P. Lepeltier, S. Peng, BSDEs with continuous coefficients and stochastic differential games, in: Pitman Research Notes in Mathematics Series, 1997, pp. 115–128.
[6] U.G. Haussmann, A Stochastic Maximum Principle for Optimal Control of Diffusions, John Wiley & Sons, Inc., 1986.
[7] P. Mannucci, Nonzero-sum stochastic differential games with discontinuous feedback, SIAM J. Control Optim. 43 (4) (2004) 1222–1233.
[8] G.J. Olsder, On open- and closed-loop bang–bang control in nonzero-sum differential games, SIAM J. Control Optim. 40 (4) (2002) 1087–1106.
[9] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Syst. Control Lett. 14 (1) (1990) 55–61.