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Algebraic geometry

Strong stability of cotangent bundles of cyclic covers [☆]*Stabilité forte du fibré cotangent des revêtements cycliques*Linguang Li ^a, Junchao Shentu ^b^a Department of Mathematics, Tongji University, Shanghai, PR China^b Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing, PR China

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ABSTRACT

Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$ of $\dim X \geq 4$ and Picard number $\rho(X) = 1$. Suppose that X satisfies $H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) = 0$ for any ample line bundle \mathcal{L} on X , and any nonnegative integers m, i, j with $0 \leq i + j < \dim X$, where $F_X : X \rightarrow X$ is the absolute Frobenius morphism. Let Y be a smooth variety obtained from X by taking hyperplane sections of $\dim \geq 3$ and cyclic covers along smooth divisors. If the canonical bundle ω_Y is ample (resp. nef), then we prove that Ω_Y is strongly stable (resp. strongly semistable) with respect to any polarization.

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R É S U M É

Soit X une variété projective lisse sur un corps algébriquement clos k de caractéristique $p > 0$ de dimension $\dim X \geq 4$ et avec nombre de Picard $\rho(X) = 1$. Supposons que X satisfasse $H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) = 0$ pour tout fibré en droite ample \mathcal{L} sur X et tous nombres entiers m, i, j tels que $0 \leq i + j < \dim X$, où $F_X : X \rightarrow X$ est le morphisme de Frobenius absolu. Soit Y une variété lisse obtenue par X en prenant des sections hyperplanes lisses de dimension ≥ 3 et des revêtements cycliques le long des diviseurs lisses. Si le fibré canonique ω_Y est ample (resp. nef), alors on montre que Ω_Y est fortement stable (resp. fortement semistable) par rapport à n'importe quelle polarisation.

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1. Introduction

An important outstanding problem in differential geometry is asking whether the tangent bundles admit Hermitian–Einstein metrics. By Kobayashi–Hitchin correspondence, this problem is related to the stability of tangent bundles. In algebraic geometry over positive characteristic, there exists another useful notion of strong stability of sheaves. X. Sun [8,9], G. Li and F. Yu [4] have showed that the strong stability of cotangent bundles has relation with the stability of Frobenius direct image of sheaves. So we would like to know which classes of varieties have strongly semistable cotangent bundles in

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E-mail addresses: LG.Lee@amss.ac.cn (L. Li), stj@amss.ac.cn (J. Shentu).

positive characteristic. However, as far as we know that there are only a few classes of varieties with strongly semistable cotangent bundles that have been found. K. Joshi [3] showed that the cotangent bundles of the general type hypersurfaces of \mathbb{P}_k^n ($n \geq 4$) are strongly stable. A. Noma [5,6] proved that any smooth weighted complete intersection X of some weak projective space with $\text{Pic}(X) \cong \mathbb{Z}$ has strongly stable cotangent bundle. Later I. Biswas [1] gave some conditions under which the cotangent bundles of complete intersections on some Fano varieties are strongly stable.

The motivation of this paper is to find new classes of varieties with strongly (semi)stable cotangent bundles in positive characteristic. T. Peternell and J. Wiśniewski [7] have studied the stability of cotangent bundles of hypersurfaces and cyclic covers over complex field. We study the strong stability of cotangent bundles of cyclic covers in positive characteristic. The main result is:

Theorem. *Let k be an algebraically closed field of characteristic $p > 0$, X an $n(\geq 4)$ -dimensional smooth projective variety of Picard number $\rho(X) = 1$ over k . Suppose that X satisfies $H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) = 0$ for any ample line bundle \mathcal{L} on X , any nonnegative integers m, i, j with $0 \leq i + j < n$. Let Y be a smooth variety obtained from X by taking hyperplane sections of $\dim \geq 3$ and cyclic covers along smooth divisors. If the canonical bundle ω_Y is ample (resp. nef), then Ω_Y is strongly stable (resp. strongly semistable).*

As an application, let X be an $n(\geq 4)$ -dimensional smooth weighted complete intersection of some weak projective space. Then the cotangent bundles of smooth ample general type (resp. non-Fano) divisors and cyclic covers along smooth ample divisors of X are strongly stable (resp. strongly semistable). (See Corollary 4.5.)

The paper is organized as follows. In Section 2 we recall some definitions and Grothendieck–Lefschetz theorem on Picard groups in arbitrary characteristic (Lemma 2.1), which is crucial for our proofs. In Section 3 we introduce the notion of Frobenius vanishing of varieties in positive characteristic (Definition 3.1), and prove that under wide condition this property is preserved under taking hypersurfaces and cyclic covers (Proposition 3.5, Theorem 3.6). In Section 4, we show that Frobenius vanishing of varieties induces the strong (semi-)stability of cotangent bundles (Theorem 4.1), and we obtain the main result of this paper (Theorem 4.4).

2. Preliminary

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety of dimension n over k with fixed ample divisors $\mathcal{H} := \{H_1, \dots, H_{n-1}\}$. The absolute Frobenius morphism $F_X : X \rightarrow X$ is induced by $\mathcal{O}_X \rightarrow \mathcal{O}_X, f \mapsto f^p$, with identity on the underlining topological space. Let \mathcal{E} be a torsion free sheaf on X , the \mathcal{H} -slope of \mathcal{E} is defined as

$$\mu_{\mathcal{H}}(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot H_1 \cdots H_{n-1}}{\text{rk}(\mathcal{E})}.$$

Then \mathcal{E} is called \mathcal{H} -stable (resp. \mathcal{H} -semistable) if $\mu_{\mathcal{H}}(\mathcal{F}) < (\text{resp. } \leq) \mu_{\mathcal{H}}(\mathcal{E})$ for any nonzero subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ with $\text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$. If $F_X^{m*}(\mathcal{E})$ is \mathcal{H} -stable (resp. \mathcal{H} -semistable) for any nonnegative integer $m \in \mathbb{N}$, then \mathcal{E} is called strongly \mathcal{H} -stable (resp. strongly \mathcal{H} -semistable).

Let \mathcal{L} be a line bundle on X , $D \in |\mathcal{L}^d|$ a smooth divisor on X for some positive integer number $d > 0$, $(p, d) = 1$. (Unless stated otherwise, we always require $(p, d) = 1$ in construction of cyclic covers.) Let $\pi : Y \rightarrow X$ be the cyclic covering over X branched along D (without confusion, we will omit mentioning \mathcal{L} in construction), then there is a smooth divisor $D' \in |\pi^*(\mathcal{L})|$ that maps isomorphically to D . Let $\Omega_X^j(\log D)$ (resp. $\Omega_Y^j(\log D')$) be the sheaf of one-forms on X (resp. Y) with logarithmic pole D (resp. D'). Then we have the isomorphism

$$\Omega_Y^j(\log D') \cong \pi^*(\Omega_X^j(\log D)),$$

which gives adjunction formula $\omega_Y \cong \pi^*(\omega_X \otimes \mathcal{L}^{d-1})$, and the following exact sequences ($1 \leq j \leq n$)

$$\begin{aligned} 0 &\longrightarrow \Omega_X^j \longrightarrow \Omega_X^j(\log D) \longrightarrow i_*\Omega_D^{j-1} \longrightarrow 0, \\ 0 &\longrightarrow \Omega_Y^j \longrightarrow \Omega_Y^j(\log D') \longrightarrow i'_*\Omega_{D'}^{j-1} \longrightarrow 0, \end{aligned}$$

where $i : D \rightarrow X$ and $i' : D' \rightarrow Y$ are the canonical embeddings.

Lemma 2.1. (See [2, Exposé XII, Corollary 3.6].) *Let X be a projective scheme over a field k , $D \subset X$ an ample Cartier divisor. Assume depth $D_x \geq 3$ for any closed points $x \in D$. Moreover, if $X \setminus D$ is regular and $H^1(D, \mathcal{O}_D(-lD)) = 0$ for any integer $l > 0$, $i = 1, 2$, then the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism.*

3. Frobenius \mathcal{H} -vanishing property

Now we introduce the notion of Frobenius \mathcal{H} -vanishing for projective varieties in positive characteristic.

Definition 3.1. Let k be an algebraically closed field of characteristic $p > 0$, and X a smooth projective variety of dimension n over k . Fix ample divisors $\mathcal{H} = \{H_1, \dots, H_{n-1}\}$ on X . A line bundle \mathcal{L} on X is called \mathcal{H} -positive (resp. \mathcal{H} -nonnegative) if

$$c_1(\mathcal{L}) \cdot H_1 \cdots H_{n-1} > (\text{resp. } \geq) 0.$$

We say that X has Frobenius \mathcal{H} -vanishing in level $m \in \mathbb{N}$ up to rank $N \in \mathbb{N}_+$ if for any \mathcal{H} -positive line bundle \mathcal{L} on X , any nonnegative integers i, j with $0 \leq i + j < N$, we have

$$H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) = 0.$$

Remark 3.2. Any ample line bundle is \mathcal{H} -positive, \mathcal{H} -positive (resp. \mathcal{H} -nonnegative) is equivalent to ample (resp. nef) if X is of Picard number 1. Hence, by [6, Proposition 2.1], any $n(\geq 3)$ -dimensional smooth weighted complete intersections of weak projective spaces have Frobenius \mathcal{H} -vanishing up to rank n in any level.

Lemma 3.3. Let k be an algebraically closed field of characteristic $p > 0$, $m \in \mathbb{N}$, $N \in \mathbb{N}_+$, X a smooth projective variety over k having Frobenius \mathcal{H} -vanishing in level m up to rank N . Let D be a smooth \mathcal{H} -nonnegative effective divisor of X . Then for any \mathcal{H} -positive line bundle \mathcal{L} on X , any nonnegative integers i, j with $0 \leq i + j < N - 1$, we have $H^i(D, F^{m*}(\Omega_D^j) \otimes (\mathcal{L}|_D)^{-1}) = 0$.

Proof. Consider the exact sequence of sheaves

$$0 \longrightarrow \Omega_X^j \otimes \mathcal{O}_X(-D) \longrightarrow \Omega_X^j \longrightarrow \Omega_X^j|_D \longrightarrow 0,$$

which is obtained by tensoring the exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with Ω_X^j ($1 \leq j \leq n$). Applying F_X^{m*} to the above sequence and tensoring with \mathcal{L}^{-1} , where \mathcal{L} is a \mathcal{H} -positive line bundle on X . Then we have exact sequence:

$$0 \longrightarrow F_X^{m*}(\Omega_X^j) \otimes \mathcal{O}_X(-D)^{p^m} \otimes \mathcal{L}^{-1} \longrightarrow F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1} \longrightarrow F_X^{m*}(\Omega_X^j|_D) \otimes \mathcal{L}^{-1} \longrightarrow 0,$$

and this let us deduce an exact sequence of cohomology groups

$$\begin{aligned} \dots &\longrightarrow H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) \longrightarrow H^i(X, F_X^{m*}(\Omega_X^j|_D) \otimes \mathcal{L}^{-1}) \\ &\longrightarrow H^{i+1}(X, F_X^{m*}(\Omega_X^j) \otimes (\mathcal{O}_X(D)^{p^m} \otimes \mathcal{L})^{-1}) \longrightarrow \dots \end{aligned}$$

Then by the Frobenius \mathcal{H} -vanishing property of X , for any nonnegative integers i, j with $0 \leq i + j < N - 1$, we have $H^i(X, F_X^{m*}(\Omega_X^j|_D) \otimes \mathcal{L}^{-1}) = 0$.

Consider the exterior power of exact sequence of cotangent-conormal sheaves

$$0 \longrightarrow \mathcal{O}_X(-D)|_D \longrightarrow \Omega_X^1|_D \longrightarrow \Omega_D^1 \longrightarrow 0,$$

we obtain the exact sequence

$$0 \longrightarrow \Omega_D^{j-1} \otimes \mathcal{O}_X(-D)|_D \longrightarrow \Omega_X^j|_D \longrightarrow \Omega_D^j \longrightarrow 0$$

for any integer $1 \leq j \leq n$. Applying F_D^{m*} to the above sequence and tensoring with line bundle $(\mathcal{L}|_D)^{-1}$, we have:

$$0 \longrightarrow F_D^{m*}(\Omega_D^{j-1}) \otimes (\mathcal{O}_X(-D)^{p^m} \otimes \mathcal{L}^{-1})|_D \longrightarrow F_D^{m*}(\Omega_X^j|_D) \otimes (\mathcal{L}|_D)^{-1} \longrightarrow F_D^{m*}(\Omega_D^j) \otimes (\mathcal{L}|_D)^{-1} \longrightarrow 0.$$

Then we have an exact sequence of cohomology groups:

$$\begin{aligned} \dots &\longrightarrow H^i(D, F_D^{m*}(\Omega_X^j|_D) \otimes (\mathcal{L}|_D)^{-1}) \longrightarrow H^i(D, F_D^{m*}(\Omega_D^j) \otimes (\mathcal{L}|_D)^{-1}) \\ &\longrightarrow H^{i+1}(D, F_D^{m*}(\Omega_D^{j-1}) \otimes (\mathcal{O}_X(-D)^{p^m} \otimes \mathcal{L}^{-1})|_D) \longrightarrow \dots \end{aligned}$$

Since $\mathcal{O}_X(D)^{p^m} \otimes \mathcal{L}$ is \mathcal{H} -positive on X , then $H^i(D, (\mathcal{O}_X(-D)^{p^m} \otimes \mathcal{L}^{-1})|_D) = 0$ for any integer $0 \leq i < N - 1$. Hence, using induction on the above sequence, we have $H^i(D, F_D^{m*}(\Omega_D^j) \otimes (\mathcal{L}|_D)^{-1}) = 0$ for any \mathcal{H} -positive line bundle \mathcal{L} on X and any nonnegative integers i, j with $0 \leq i + j < N - 1$. \square

Corollary 3.4. Let k be an algebraically closed field, X an $n(\geq 4)$ -dimensional smooth projective variety over k . Suppose that X has Frobenius \mathcal{H} -vanishing in level 0 up to rank $N(\geq 3)$. Let D be a smooth ample effective divisor of X . Then the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism.

Proof. By Lemma 3.3, we have $H^i(D, \mathcal{O}_D(-nD)) = 0$ for any integer $n > 0, i = 1, 2$. Hence by Lemma 2.1, the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism. \square

Proposition 3.5. Let k be an algebraically closed field of characteristic $p > 0, m \in \mathbb{N}, X$ an $n(\geq 4)$ -dimensional smooth projective variety over k of Picard number $\rho(X) = 1$. If X has Frobenius \mathcal{H} -vanishing in level m up to rank $N(\geq 3)$. Then any smooth ample effective divisors of X have Frobenius vanishing with respect to any polarization in level m up to rank $N - 1$.

Proof. Since $\rho(X) = 1$. All \mathcal{H} -positive line bundles are ample, and all ample line bundles are numerical equivalent up to a positive scalar. By Corollary 3.4, D is also of Picard number $\rho(D) = 1$. Therefore, any line bundle on D is of the form $\mathcal{L}|_D$, where \mathcal{L} is a line bundle on X . But \mathcal{L} is ample on X if and only if $\mathcal{L}|_D$ is ample on D . Hence this proposition follows from Lemma 3.3. \square

Theorem 3.6. Let k be an algebraically closed field of characteristic $p > 0, m \in \mathbb{N}, N \in \mathbb{N}_+$, and X an $n(\geq 4)$ -dimensional smooth projective variety over k, D a smooth ample divisor on X , and $\pi : Y \rightarrow X$ a cyclic cover of X branched along D . Suppose that X has Frobenius \mathcal{H} -vanishing in level m up to rank $N(\geq 3)$. Then Y also has Frobenius $\pi^*(\mathcal{H})$ -vanishing in level m up to rank N .

Proof. By construction of the cyclic cover, there exists a smooth divisor $i' : D' \rightarrow Y$ maps isomorphically to D . This implies $H^i(D, \mathcal{O}_D(-iD)) \cong H^i(D', \mathcal{O}_{D'}(-iD'))$. Moreover, by Corollary 3.4, the commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{i'} & Y \\ \cong \downarrow & & \downarrow \pi \\ D & \xrightarrow{i} & X, \end{array}$$

lets us deduce the following commutative diagram

$$\begin{array}{ccc} \text{Pic}(D') & \xleftarrow{\cong} & \text{Pic}(Y) \\ \cong \uparrow & & \uparrow \pi^* \\ \text{Pic}(D) & \xleftarrow{\cong} & \text{Pic}(X). \end{array}$$

This implies the homomorphism $\pi^* : \text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism. Let $\mathcal{L} \in \text{Pic}(X)$, then \mathcal{L} is \mathcal{H} -positive (resp. \mathcal{H} -nonnegative) on X if and only if $\pi^*\mathcal{L}$ is $\pi^*\mathcal{H}$ -positive (resp. nonnegative) on Y , where $\pi^*(\mathcal{H}) := \{\pi^*(H_1), \dots, \pi^*(H_{n-1})\}$.

Applying F_X^{m*} to the following sequence and tensoring with \mathcal{L}^{-1} ,

$$0 \rightarrow \Omega_X^j \rightarrow \Omega_X^j(\log D) \rightarrow i_*\Omega_D^{j-1} \rightarrow 0,$$

where \mathcal{L} is an \mathcal{H} -positive line bundle on X . We have the exact sequence:

$$0 \rightarrow F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1} \rightarrow F_X^{m*}(\Omega_X^j(\log D)) \otimes \mathcal{L}^{-1} \rightarrow F_X^{m*}(i_*\Omega_D^{j-1}) \otimes \mathcal{L}^{-1} \rightarrow 0.$$

This lets us deduce the exact sequence of cohomology groups:

$$\dots \rightarrow H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) \rightarrow H^i(X, F_X^{m*}(\Omega_X^j(\log D)) \otimes \mathcal{L}^{-1}) \rightarrow H^i(X, F_X^{m*}(i_*\Omega_D^{j-1}) \otimes \mathcal{L}^{-1}) \rightarrow \dots$$

Notice that $F_X^{m*}i_* \cong i_*F_D^{m*}$, there exists a natural isomorphism:

$$F_X^{m*}(i_*\Omega_D^{j-1}) \otimes \mathcal{L}^{-1} \cong i_*(F_D^{m*}(\Omega_D^{j-1}) \otimes (\mathcal{L}|_D)^{-1}).$$

Then, by Lemma 3.3, for any nonnegative integers i, j with $0 \leq i + j < N$, we have $H^i(X, F_X^{m*}(\Omega_X^j(\log D)) \otimes \mathcal{L}^{-1}) = 0$. Hence $H^i(X, F_X^{m*}(\Omega_X^j) \otimes \mathcal{L}^{-1}) = 0$.

From the isomorphism $\Omega_Y^j(\log D') \cong \pi^*(\Omega_X^j(\log D))$, we have the isomorphisms:

$$F_Y^{m*}(\Omega_Y^j(\log D')) \cong F_Y^{m*}(\pi^*(\Omega_X^j(\log D))) \cong \pi^*(F_X^{m*}(\Omega_X^j(\log D))).$$

As $\pi_*(\mathcal{O}_Y) \cong \bigoplus_{0 \leq i < d} \mathcal{O}(D)^{-i}$, so by projective formula, we get the isomorphism:

$$\pi_*(F_Y^{m*}(\Omega_Y^j(\log D'))) \cong F_X^{m*}(\Omega_X^j(\log D)) \otimes \bigoplus_{0 \leq i < d} \mathcal{O}(D)^{-i}.$$

Since $\pi : Y \rightarrow X$ is an affine morphism, by projective formula, we have:

$$H^i(Y, F_Y^{m*}(\Omega_Y^j(\log D'))) \otimes \pi^*(\mathcal{L}^{-1}) \cong H^i\left(Y, F_X^{m*}(\Omega_X^j(\log D)) \otimes \left(\bigoplus_{0 \leq i < d} \mathcal{O}(D)^{-i}\right) \otimes \mathcal{L}^{-1}\right) = 0.$$

Consider the following exact sequence on Y ($1 \leq j \leq n$):

$$0 \longrightarrow \Omega_Y^j \longrightarrow \Omega_Y^j(\log D') \longrightarrow i'_* \Omega_{D'}^{j-1} \longrightarrow 0.$$

Applying F_Y^{m*} to the above sequence and tensoring with $\pi^*(\mathcal{L}^{-1})$, we get the exact sequence

$$0 \longrightarrow F_Y^{m*}(\Omega_Y^j) \otimes \pi^*(\mathcal{L}^{-1}) \longrightarrow F_Y^{m*}(\Omega_Y^j(\log D')) \otimes \pi^*(\mathcal{L}^{-1}) \longrightarrow F_Y^{m*}(i'_* \Omega_{D'}^{j-1}) \otimes \pi^*(\mathcal{L}^{-1}) \longrightarrow 0,$$

Notice that $F_Y^{m*}i'_* \cong i'_* F_{D'}^{m*}$, there exists a natural isomorphism

$$F_Y^{m*}(i'_* \Omega_{D'}^{j-1}) \otimes \pi^*(\mathcal{L}^{-1}) \cong i_*(F_{D'}^{m*}(\Omega_{D'}^{j-1}) \otimes \pi^*(\mathcal{L}^{-1})|_{D'}).$$

Taking cohomology, we have:

$$\begin{aligned} \dots &\longrightarrow H^{i-1}(D', F_{D'}^{m*}(\Omega_{D'}^{j-1}) \otimes \pi^*(\mathcal{L}^{-1})|_{D'}) \longrightarrow H^i(Y, F_Y^{m*}(\Omega_Y^j) \otimes \pi^*(\mathcal{L}^{-1})) \\ &\longrightarrow H^i(Y, F_Y^{m*}(\Omega_Y^j(\log D')) \otimes \pi^*(\mathcal{L}^{-1})) \longrightarrow \dots \end{aligned}$$

Since D' maps isomorphically to D with commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{i'} & Y \\ \cong \downarrow & & \downarrow \pi \\ D & \xrightarrow{i} & X \end{array}$$

therefore, there is the isomorphism

$$H^{i-1}(D', F_{D'}^{m*}(\Omega_{D'}^{j-1}) \otimes \pi^*(\mathcal{L}^{-1})|_{D'}) \cong H^{i-1}(D, F_D^{m*}(\Omega_D^{j-1}) \otimes (\mathcal{L}|_D)^{-1}).$$

Then, by Lemma 3.3, for any nonnegative integers i, j with $0 \leq i + j < N$, we have $H^i(Y, F_Y^{m*}(\Omega_Y^j) \otimes \pi^*(\mathcal{L}^{-1})) = 0$. This completes the proof of this theorem. \square

4. Strong stability of cotangent bundles

Frobenius \mathcal{H} -vanishing property has closed relation with the strong stability of cotangent bundles in positive characteristic, at least for smooth projective varieties of Kodaira dimension ≥ 0 .

Proposition 4.1. *Let k be an algebraically closed field of characteristic $p > 0$, $m \in \mathbb{N}$, and X a smooth projective variety over k . Suppose that X has Frobenius \mathcal{H} -vanishing in level m up to rank $\dim X$ and ω_X is \mathcal{H} -positive (resp. \mathcal{H} -nonnegative). Then $F_X^{m*}(\Omega_X)$ is \mathcal{H} -stable (resp. \mathcal{H} -semistable).*

Proof. This is a classical argument. Assume that $F_X^{m*}(\Omega_X)$ is not \mathcal{H} -stable (resp. \mathcal{H} -semistable), then there is a reflexive subsheaf $\mathcal{E} \subseteq F_X^{m*}(\Omega_X)$ of rank $j (< \dim X)$ such that $\mu_{\mathcal{H}}(\mathcal{E}) \geq$ (resp. $>$) $\mu_{\mathcal{H}}(F_X^{m*}(\Omega_X))$. This induces a nontrivial homomorphism $\det(\mathcal{E}) \rightarrow F_X^{m*}(\Omega_X^j)$. Since ω_X is \mathcal{H} -positive (resp. \mathcal{H} -nonnegative), we have $\det(\mathcal{E})$ is \mathcal{H} -positive. This contradicts the Frobenius \mathcal{H} -vanishing assumption on X . Hence, $F_X^{m*}(\Omega_X)$ is \mathcal{H} -stable (resp. \mathcal{H} -semistable). \square

Remark 4.2. If $\kappa(X) > 0$ (resp. ≥ 0), then ω_X is \mathcal{H} -positive (resp. \mathcal{H} -nonnegative).

Corollary 4.3. *Let k be an algebraically closed field of characteristic $p > 0$, $m \in \mathbb{N}$, and X an $n (\geq 4)$ -dimensional smooth projective variety over k , $\pi : Y \rightarrow X$ a cyclic cover of X branched along a smooth ample divisor D . Suppose that X has Frobenius \mathcal{H} -vanishing in level m up to rank n , and ω_Y is $\pi^*(\mathcal{H})$ -positive (resp. $\pi^*(\mathcal{H})$ -nonnegative). Then $F_X^{m*}(\Omega_Y)$ is $\pi^*(\mathcal{H})$ -stable (resp. $\pi^*(\mathcal{H})$ -semistable).*

Proof. It is obvious by Theorem 3.6 and Proposition 4.1. \square

Combining all results above, we obtain the main result of the paper.

Theorem 4.4. *Let k be an algebraically closed field of characteristic $p > 0$, X an $n (\geq 4)$ -dimensional smooth projective variety of Picard number $\rho(X) = 1$ over k . Suppose that X has Frobenius \mathcal{H} -vanishing of rank n in any level. Let Y be a smooth variety obtained from X by taking hyperplane sections of $\dim \geq 3$ and cyclic covers along smooth divisors. If the canonical bundle ω_Y is ample (resp. nef), then Ω_Y is strongly stable (resp. strongly semistable) with respect to any polarization.*

Proof. By [Corollary 3.4](#) and proof of [Theorem 3.6](#), we have $\rho(Y) = 1$. Hence, any polarization are numerical equivalent up to a positive scalar. Combining [Proposition 3.5](#), [Theorem 3.6](#) and [Proposition 4.1](#), we get the strong stability of Ω_Y with respect to any polarization. \square

Corollary 4.5. *Let k be an algebraically closed field of characteristic $p > 0$, and X an $n(\geq 4)$ -dimensional smooth weighted complete intersection of a weak projective space. Let Y be a smooth variety obtained from X by taking hyperplane sections of $\dim \geq 3$ and cyclic covers along smooth divisors. If the canonical bundle ω_Y is ample (resp. nef), then Ω_Y is strongly stable (resp. strongly semistable) with respect to any polarization.*

Proof. It easily follows from [Theorem 4.4](#) and [Remark 3.2](#). \square

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References

- [1] I. Biswas, Tangent bundle of a complete intersection, *Trans. Amer. Math. Soc.* 362 (6) (2010) 3149–3160.
- [2] A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), *Adv. Stud. Pure Math.*, vol. 2, North-Holland Publishing Co./Masson & Cie, Amsterdam/Paris, 1968.
- [3] K. Joshi, Kodaira–Akizuki–Nakano vanishing: a variant, *Bull. Lond. Math. Soc.* 32 (2) (2000) 171–176.
- [4] G. Li, F. Yu, Instability of truncated symmetric powers of sheaves, *J. Algebra* 386 (2013) 176–189.
- [5] A. Noma, Stability of Frobenius pull-backs of tangent bundles and generic injectivity of Gauss maps in positive characteristic, *Compos. Math.* 106 (1997) 61–70.
- [6] A. Noma, Stability of Frobenius pull-backs of tangent bundles of weighted complete intersections, *Math. Nachr.* 221 (2001) 87–93.
- [7] T. Peternell, J. Wiśniewski, On the stability of tangent bundles of Fano manifolds, *J. Algebr. Geom.* 4 (1995) 363–384.
- [8] X. Sun, Direct images of bundles under Frobenius morphisms, *Invent. Math.* 173 (2) (2008) 427–447.
- [9] X. Sun, Stability of sheaves of locally closed and exact forms, *J. Algebra* 324 (2010) 1471–1482.