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Transmutation operators associated with an integro-differential operator on the real line and certain of their applications

Opérateurs de transmutation associés à un opérateur intégro-différentiel sur la droite réelle et certaines de leurs applications

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ABSTRACT

We consider a singular integro-differential operator Λ on the real line. We build transmutation operators of Λ and its dual $\widetilde{\Lambda}$ into the first derivative operator $\mathrm{d}/\mathrm{d}x$. Using these transmutation operators, we develop a new commutative harmonic analysis on the real line corresponding to the operator Λ.

RÉSUMÉ

Nous considérons un opérateur integro-différentiel singulier Λ sur la droite réelle. Nous construisons une paire de transformations intégrales qui transmutent Λ et son dual $\widetilde{\Lambda}$ en l'opérateur $\mathrm{d}/\mathrm{d}x$. En utilisant les propriétés de ces opérateurs de transmutation, on définit une nouvelle analyse harmonique sur $\mathbb{R}$ correspondant à l'opérateur Λ.

1. Notations

We denote by $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. Recall that each function f in $\mathcal{E}(\mathbb{R})$ may be decomposed uniquely into the sum $f = f_\mathrm{e} + f_\mathrm{o}$, where the even part f_e is defined by $f_\mathrm{e}(x) = (f(x) + f(-x))/2$ and the odd part f_o by $f_\mathrm{o}(x) = (f(x) - f(-x))/2$. $\mathcal{E}_\mathrm{e}(\mathbb{R})$ (resp. $\mathcal{E}_\mathrm{o}(\mathbb{R})$) stands for the subspace of $\mathcal{E}(\mathbb{R})$ consisting of even (resp. odd) functions. For $a > 0$, $\mathcal{D}_a(\mathbb{R})$ designates the space of C^∞ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$. Put $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology. $\mathcal{D}_\mathrm{e}(\mathbb{R})$ (resp. $\mathcal{D}_\mathrm{o}(\mathbb{R})$) denotes the subspace of $\mathcal{D}(\mathbb{R})$ consisting of even (resp. odd) functions. For $a > 0$, let $\mathbf{H}_a$ be the space of entire, rapidly decreasing functions of exponential type a. Put $\mathbf{H} = \bigcup_{a>0} \mathbf{H}_a$, endowed with the inductive limit topology. Let $\mathcal{I}$ (resp. $\mathcal{J}$) denotes the map defined on $\mathcal{E}_\mathrm{e}(\mathbb{R})$ (resp. $\mathcal{D}_\mathrm{o}(\mathbb{R})$) by $\mathcal{I}h(x) = \frac{1}{A(x)} \int_0^x h(t)A(t)\mathrm{d}t$ (resp. $\mathcal{J}h(x) = \int_{-\infty}^x h(t)\mathrm{d}t$).

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2. Transmutation operators

In [4] we have considered the first-order singular differential-difference operator

$$\Lambda_0 f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

B being a positive C^∞ even function on $\mathbb{R}$. We have exploited a pair of transmutation operators between Λ_0 and the first derivative operator $\mathrm{d}/\mathrm{d}x$, to initiate a quite new harmonic analysis on the real line tied to Λ_0, in which several analytic structures on $\mathbb{R}$ were generalized. The key role in our investigation was played by the second-order differential operator

$$\Delta_0 f(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}f}{\mathrm{d}x},$$

which is linked to Λ_0 via the relationship

$$\Lambda_0^2 f = \Delta_0 f, \quad \text{for all } f \in \mathcal{E}_\mathrm{e}(\mathbb{R}).$$

Put

$$\Delta = \Delta_0 + q,$$

where q is a real-valued C^∞ even function on $\mathbb{R}$. The motivation of the present paper was to look for an integro-differential operator of the form

$$\Lambda = \Lambda_0 + M(x)\int_{-x}^{x} f(t)N(t)\mathrm{d}t$$

(M and N being two even functions) such that

$$\Lambda^2 f = \Delta f, \quad \text{for all } f \in \mathcal{E}_\mathrm{e}(\mathbb{R}). \tag{1}$$

A straightforward calculation shows that (1) is equivalent to

$$(2MN - q)f + \frac{2}{A}(AM)'\int_0^x fN\mathrm{d}t = 0,$$

for all $f \in \mathcal{E}_\mathrm{e}(\mathbb{R})$. The easiest choice was

$$AM = 1 \quad \text{and} \quad 2MN - q = 0,$$

that is,

$$\Lambda = \Lambda_0 + \frac{1}{A(x)}\int_0^x \left(\frac{f(t)+f(-t)}{2}\right)q(t)A(t)\mathrm{d}t.$$

The objective of this work is to establish for Λ results similar to those obtained for Λ_0 in [4]. This objective is achieved by using the crucial identity (1) and some basic facts about the differential operator Δ. Recall that Lions [2] has constructed an automorphism $\mathcal{X}$ of $\mathcal{E}_\mathrm{e}(\mathbb{R})$ satisfying

$$\mathcal{X}\frac{\mathrm{d}^2}{\mathrm{d}x^2}f = \Delta\mathcal{X}f \quad \text{and} \quad \mathcal{X}f(0) = f(0) \quad \text{for all } f \in \mathcal{E}_\mathrm{e}(\mathbb{R}).$$

The construction of the Lions operator $\mathcal{X}$ was aimed at allowing the resolution of certain mixed value problems. Later, Trimèche [5] has obtained for the Lions operator $\mathcal{X}$ the following integral representation:

$$\mathcal{X}f(x) = \int_0^{|x|} \mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \neq 0,\ f \in \mathcal{E}_\mathrm{e}(\mathbb{R}), \tag{2}$$

where $\mathcal{K}(x,\cdot):\mathbb{R}\to\mathbb{R}$ is an even continuous function on $]-|x|,|x|[$, with support in $[-|x|,|x|]$. Moreover, he proved that the integral transform

$$ {}^{\mathrm{t}}\mathcal{X}f(y)=\int_{|y|}^{\infty}\mathcal{K}(x,y)f(x)A(x)\mathrm{d}x,\quad y\in\mathbb{R}, \tag{3}$$

is an automorphism of $\mathcal{D}_{\mathrm{e}}(\mathbb{R})$ satisfying the intertwining relation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}{}^{\mathrm{t}}\mathcal{X}f={}^{\mathrm{t}}\mathcal{X}\Delta f,\quad f\in\mathcal{D}_{\mathrm{e}}(\mathbb{R}).$$

We claim the next statements.

Theorem 2.1. *The map*

$$Vf=\mathcal{X}(f_{\mathrm{e}})+\mathcal{I}\mathcal{X}\frac{\mathrm{d}}{\mathrm{d}x}(f_{\mathrm{o}}) \tag{4}$$

is the only automorphism of $\mathcal{E}(\mathbb{R})$ *satisfying*

$$V\frac{\mathrm{d}}{\mathrm{d}x}f=\Lambda Vf\quad\textit{and}\quad Vf(0)=f(0)\quad\textit{for all } f\in\mathcal{E}(\mathbb{R}).$$

Theorem 2.2. *The map*

$$ {}^{\mathrm{t}}Vf={}^{\mathrm{t}}\mathcal{X}(f_{\mathrm{e}})+\frac{\mathrm{d}}{\mathrm{d}x}{}^{\mathrm{t}}\mathcal{X}\mathcal{J}(f_{\mathrm{o}}) \tag{5}$$

is an automorphism of $\mathcal{D}(\mathbb{R})$ *satisfying the intertwining relation*

$$\frac{\mathrm{d}}{\mathrm{d}x}{}^{\mathrm{t}}Vf={}^{\mathrm{t}}V\widetilde{\Lambda}f,\quad f\in\mathcal{D}(\mathbb{R}),$$

$\widetilde{\Lambda}$ *being the dual operator of* Λ *defined by*

$$\widetilde{\Lambda}f(x)=\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right)+q(x)\int_{-\infty}^{x}\left(\frac{f(t)-f(-t)}{2}\right)\mathrm{d}t.$$

Remark 2.1. (i) If $A(x)=|x|^{2\alpha+1}$ and $q(x)=0$, then the integro-differential operator Λ reduces to the so-called Dunkl operator with parameter $\alpha+1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Moreover, $V(f)(x)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}f(tx)(1-t^2)^{\alpha-1/2}(1+t)\,\mathrm{d}t$ (see [1]).

(ii) The integro-differential operators Λ and $\widetilde{\Lambda}$ are connected by the integral formula: $\int_{\mathbb{R}}\Lambda f(x)g(x)A(x)\mathrm{d}x=-\int_{\mathbb{R}}f(x)\widetilde{\Lambda}g(x)A(x)\mathrm{d}x$, which is true for every $f\in\mathcal{E}(\mathbb{R})$ and $g\in\mathcal{D}(\mathbb{R})$.

(iii) The integral transform V (resp. ${}^{\mathrm{t}}V$) is said to be a transmutation operator between Λ (resp. $\widetilde{\Lambda}$) and the first derivative operator $\mathrm{d}/\mathrm{d}x$ on the space $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{D}(\mathbb{R})$).

(iv) The integral transforms V and ${}^{\mathrm{t}}V$ are dual by virtue of the relation: $\int_{\mathbb{R}}Vf(x)g(x)A(x)\mathrm{d}x=\int_{\mathbb{R}}f(y)\,{}^{t}Vg(y)\mathrm{d}y$, valid for any $f\in\mathcal{E}(\mathbb{R})$ and $g\in\mathcal{D}(\mathbb{R})$.

(v) A combination of (2), (3), (4) and (5) yields

$$Vf(x)=\int_{-|x|}^{|x|}K(x,y)f(y)\mathrm{d}y\quad\left(\text{resp. } {}^{\mathrm{t}}Vf(y)=\int_{|x|\geq|y|}K(x,y)f(x)A(x)\mathrm{d}x\right)$$

with

$$K(x,y)=\frac{1}{2}\mathcal{K}(x,y)-\frac{\mathrm{sgn}(x)}{2A(x)}\frac{\partial}{\partial y}\left(\int_{|y|}^{|x|}\mathcal{K}(t,y)A(t)\mathrm{d}t\right).$$

3. Generalized Fourier transform

The generalized Fourier transform of a function $f \in \mathcal{D}(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)\mathrm{d}x, \quad \lambda \in \mathbb{C},$$

where $\Phi_{-\lambda}(x) = V(\mathrm{e}^{-\mathrm{i}\lambda\cdot})(x)$. The Dunkl transform with parameter $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ is a particular case of $\mathcal{F}$ corresponding to $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$. The generalized Fourier transform $\mathcal{F}$ is linked to the classical Fourier transform $^\wedge$ on $\mathbb{R}$ via the relation:

$$\mathcal{F}(f)(\lambda) = \left({}^{\mathrm{t}}Vf\right)^{\wedge}(\lambda), \quad f \in \mathcal{D}(\mathbb{R}). \tag{6}$$

Furthermore, we have the decomposition:

$$\mathcal{F}_{\Lambda}(f)(\lambda) = \mathcal{F}_{\Delta}(f_{\mathrm{e}})(\lambda) + \mathrm{i}\lambda\mathcal{F}_{\Delta}\mathcal{J}(f_{\mathrm{o}})(\lambda), \tag{7}$$

where $\mathcal{F}_{\Delta}$ stands for the Fourier transform related to the differential operator Δ, defined on $\mathcal{D}_{\mathrm{e}}(\mathbb{R})$ by $\mathcal{F}_{\Delta}(f)(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{\lambda}(x)A(x)\mathrm{d}x$; φ_{λ} being the solution of the differential equation $\Delta u = -\lambda^2 u$, $u(0) = 1$ (see [5]). From (6) and the classical Paley–Wiener theorem, we deduce the next theorem:

Theorem 3.1 *(Paley–Wiener). The generalized Fourier transform $\mathcal{F}$ is an isomorphism from $\mathcal{D}(\mathbb{R})$ onto* **H**. *More precisely, $f \in \mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}(f) \in \mathbf{H}_a$.*

Combining (7) and [5, Chapter 9], we establish for $\mathcal{F}$ the following two standard results:

Theorem 3.2 *(Inversion formula). For all $f \in \mathcal{D}(\mathbb{R})$,*

$$f(x) + \mathcal{I}(q\mathcal{J}f_{\mathrm{o}})(x) = \int_{\mathbb{R}} \mathcal{F}(f)(\lambda)\Phi_{\lambda}(x)\mathrm{d}\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}(f)(\mathrm{i}\lambda)\Phi_{i\lambda}(x)\mathrm{d}\mu_2(\lambda),$$

where μ_1 is an even positive tempered measure on $\mathbb{R}$, and μ_2 is an even positive measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \mathrm{e}^{a|y|}\mathrm{d}\mu_2(y) < \infty, \quad \textit{for all } a > 0.$$

Theorem 3.3 *(Parseval formula). For all $f, g \in \mathcal{D}(\mathbb{R})$,*

$$\int_{\mathbb{R}} f(y)g(-y)A(y)\mathrm{d}y + \int_{\mathbb{R}} q(y)\mathcal{J}f_{\mathrm{o}}(y)\mathcal{J}g_{\mathrm{o}}(y)A(y)\mathrm{d}y = \int_{\mathbb{R}} \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda)\mathrm{d}\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}(f)(\mathrm{i}\lambda)\mathcal{F}(g)(\mathrm{i}\lambda)\mathrm{d}\mu_2(\lambda),$$

μ_1 and μ_2 being as in Theorem 3.2.

Remark 3.1. If $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, then $\mathrm{d}\mu_1(\lambda) = 2^{-(2\alpha+2)}(\Gamma(\alpha+1))^{-2}|\lambda|^{2\alpha+1}\mathrm{d}\lambda$ and $\mu_2 = 0$.

4. Generalized translation operators

With the help of the transmutation operator V, we introduce in $\mathcal{E}(\mathbb{R})$ generalized translation operators T^x, $x \in \mathbb{R}$, defined by:

$$T^x f(y) = V_x V_y\left[V^{-1}f(x+y)\right], \quad y \in \mathbb{R}.$$

The basic properties of the T^x, $x \in \mathbb{R}$, are provided by the following statement:

Theorem 4.1. *(i) For all $x \in \mathbb{R}$, T^x is a linear bounded operator from $\mathcal{E}(\mathbb{R})$ into itself; the function $x \mapsto T^x$ is C^∞.*

(ii) We have: $T^0 = \text{identity}$, $T^xT^y = T^yT^x$, $\Lambda T^x = T^x\Lambda$.

(iii) For all $f \in \mathcal{E}(\mathbb{R})$, $T^x f(y) = T^y f(x)$.

(iv) For each $\lambda \in \mathbb{C}$, we have the product formula: $T^x(\Phi_\lambda)(y) = \Phi_\lambda(x)\Phi_\lambda(y)$.

(v) For all $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$, we have: $\int_{\mathbb{R}} T^x f(y)g(y)A(y)\mathrm{d}y = \int_{\mathbb{R}} f(y){}^{\mathrm{t}}T^x g(y)A(y)\mathrm{d}y$, where ${}^{\mathrm{t}}T^x g(y) = V_x({}^{\mathrm{t}}V^{-1})_y[{}^{\mathrm{t}}Vg(y-x)]$.

(vi) Let f be in $\mathcal{D}_a(\mathbb{R})$, $a > 0$. Then for all $x \in \mathbb{R}$, ${}^{\mathrm{t}}T^x f$ is an element of $\mathcal{D}_{a+|x|}(\mathbb{R})$ and $\mathcal{F}({}^{\mathrm{t}}T^x f)(\lambda) = \Phi_{-\lambda}(x)\mathcal{F}f(\lambda)$.

Let $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$. The generalized convolution product of f and g is the function $f\#g \in \mathcal{E}(\mathbb{R})$ defined by:

$$f\#g(x) = \int_{\mathbb{R}} {}^{\mathrm{t}}T^{y} f(x) g(y) A(y) \mathrm{d}y, \quad x \in \mathbb{R}.$$

Theorem 4.2. *(i) Let $f \in \mathcal{D}_a(\mathbb{R})$ and $g \in \mathcal{D}_b(\mathbb{R})$. Then $f\#g \in \mathcal{D}_{a+b}(\mathbb{R})$ and $\mathcal{F}(f\#g)(\lambda) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda)$.*
(ii) For all $f, g \in \mathcal{D}(\mathbb{R})$, we have ${}^{\mathrm{t}}V(f\#g) = {}^{\mathrm{t}}Vf * {}^{\mathrm{t}}Vg$, where $*$ stands for the usual convolution on $\mathbb{R}$.
(iii) For all $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$, we have $V(f * {}^{\mathrm{t}}Vg) = V(f)\#g$.

Remark 4.1. It is pointed out that all the results obtained in [4] may be recovered from those stated in the present work by simply taking $q = 0$. As for Lions operators [3], it is believed that our transmutation operators will be of great utility in the study of integro-differential problems, and will lead to generalizations of various analytic structures on the real line.

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