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A note on a characterization of generalized quaternion 2-groups ☆

Caractérisation des 2-groupes de quaternions généralisés

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ABSTRACT

In this note, we answer an open problem posed in M. Tărnăceanu (2010) [5], and obtain that the generalized quaternion 2-groups are the unique finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.

RÉSUMÉ

Répondant à une question de M. Tărnăceanu (2010) [5], nous montrons que les 2-groupes de quaternions généralisés sont les seuls groupes finis non cycliques dont le treillis des classes de conjugaison de sous-groupes cycliques admet un point clivant.

1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. A proper nontrivial subgroup H of G is called a *breaking point* for $L(G)$ if and only if

$$\text{for every } X \in L(G), \quad \text{we have } X \leq H \text{ or } X \geq H.$$

Such subgroups have been studied in paper [1]. In paper [5], the author extended the concept to the poset of cyclic subgroups of a finite group, denoted by $C(G)$, and proved that the generalized quaternion 2-groups are the only finite noncyclic groups whose posets of cyclic subgroups have breaking points. Further, also in the paper [5], the author generalized the concept again and extended it to the poset of conjugacy classes of cyclic subgroups of G, denoted by $\overline{C}(G) = \{[H] | H \in C(G)\}$. It seems that $[H]$ being a breaking point of $\overline{C}(G)$ is weaker than the condition where H is a breaking point of $C(G)$. And the author [5] remarked that for a finite p-group G, the poset $\overline{C}(G)$ possesses breaking points if and only if G is either a cyclic p-group of order at least p^2 or a generalized quaternion 2-group, and that for an arbitrary finite group G, the problem of

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characterizing the existence and the uniqueness of breaking points of $\overline{C}(G)$ remains still open. In this note, we will answer this open problem. Our main theorem proves that the generalized quaternion 2-groups exhaust all finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.

Theorem 1.1. *Let G be a finite group. Then the poset $\overline{C}(G)$ possesses breaking points if and only if G is either a cyclic p-group of order at least p^2 or a generalized quaternion 2-group.*

Further, by Theorem 1.1 of [5], we can obtain that for a finite group G, two conditions, i.e. the poset $C(G)$ has breaking points and the poset $\overline{C}(G)$ has breaking points, are equivalent.

The notation and terminologies are standard in this note, and the reader is referred to [3] for group theory and [4] for subgroup lattice theory if necessary.

2. The proof of Theorem 1.1

To prove the Theorem 1.1, we cite the following crucial Theorem 1 of paper [2], proved by using the classification of finite simple groups.

Theorem 2.1. *Let G be a finite group acting transitively on a set Ω with $|\Omega| > 1$. Then there exists a prime r and an r-element $g \in G$ such that g acts without fixed points on Ω.*

As we all know, a finite group can be generated by the representatives of all its conjugacy classes. Using Theorem 2.1, we can generalize this conclusion and obtain that a finite group can be generated by the representatives of all its conjugacy classes of prime power order elements. This is the following lemma.

Lemma 2.2. *Let G be a finite group and H be a subgroup of G. Suppose that for each prime power order element, there exists some element $g \in G$ such that $x^g \in H$. Then $G = H$.*

Proof. By the way of contradiction, assume that H is a proper nontrivial subgroup of G. Let Ω be the set of right cosets of H in G. Then $\Omega = \{Hg | g \in G\}$ and $|\Omega| > 1$. Considering the action of G on the set Ω, we have that G acts transitively on Ω, and so for each element $Hg \in \Omega$, the stabilizer G_{Hg} of Hg is equal to H^g. By hypothesis, since every element of prime power order of G is conjugate to an element of H, we get that each element of prime power order of G has a fixed point on the set Ω. On the other hand, in view of $|\Omega| > 1$ and G acting transitively on Ω, by Theorem 2.1 we have that there exists a prime power order element that acts fixed-point-free on Ω. Hence a contradiction is derived, and thus $G = H$. $\square$

For convenience, we put a remark of the paper [5] about a finite p-group as the next lemma.

Lemma 2.3. *Let G be a finite p-group. Then the poset $\overline{C}(G)$ possesses breaking points if and only if G is either a cyclic p-group of order at least p^2 or a generalized quaternion 2-group.*

Proof of Theorem 1.1. Since the necessity is obvious, it is enough to prove the sufficiency. And by Lemma 2.3, it is sufficient to prove that G must be a group of prime power order.

Assume that G is not a group of prime power order. Then $|\pi(G)| > 1$, that is, $|G|$ has at least two distinct prime divisors. Let $[H]$ be a breaking point of $\overline{C}(G)$. By the definition of $\overline{C}(G)$, we have that for any $X \in C(G)$, there exists an element $g \in G$ satisfying that $X^g \leq H$ or $X^g \geq H$. It follows that $|H|$ has more than two distinct prime divisors. Let $p \in \pi(G)$ and K be a cyclic p-subgroup of G. Then there exists an element $g \in G$ such that $K^g \leq H$ or $K^g \geq H$. Since $|\pi(H)| > 1$, we get $K^g \leq H$. Hence for every prime power order element $x \in G$, x is conjugate to an element of H. By Lemma 2.2, we have $G = H$, a contradiction with $H < G$. Therefore, G is a group of prime power order. $\square$

By the results of Theorem 1.1, we easily obtain the following two corollaries.

Corollary 2.4. *Let G be a finite group. Then the poset $\overline{C}(G)$ possesses a unique breaking point if and only if G is either a cyclic p-group of order p^2 or a generalized quaternion 2-group.*

Corollary 2.5. *The generalized quaternion 2-groups are the only finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.*

Comparing Theorem 1.1 with Theorem 1.1 of [5], we obtain the following corollary.

Corollary 2.6. *Let G be a finite group. Then the poset $C(G)$ has breaking points if and only if the poset $\overline{C}(G)$ has breaking points.*

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