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Partial differential equations

Locally periodic thin domains with varying period

Domaines minces localement périodiques avec une période variable

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ABSTRACT

We analyze the behavior of the solutions of the Laplace equation with Neumann boundary conditions in a thin domain with a highly oscillatory behavior. The oscillations are locally periodic in the sense that both the amplitude and the period of the oscillations may not be constant and actually they vary in space. We obtain the asymptotic homogenized limit and provide some correctors. To accomplish this goal, we extend the unfolding operator method to the locally periodic case. The main ideas of this extension may be applied to other cases like perforated domains or reticulated structures, which are locally periodic with not necessarily a constant period.

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RÉSUMÉ

Nous analysons le comportement des solutions de l'équation de Laplace dans un domaine mince à frontière fortement oscillante. Les oscillations sont localement périodiques dans le sens où l'amplitude et la période ne sont pas nécessairement constantes, puisqu'elles varient en espace. Nous obtenons le problème limite homogénéisé et montrons les résultats des correcteurs. Pour atteindre notre objectif, nous étendons la méthode de l'opérateur d'éclatement à des cas localement périodiques. Les idées principales de cette extension peuvent être appliquées à d'autres cas, comme par exemple à des domaines perforés ou à des structures réticulées localement périodiques dont la période n'est pas forcément constante.

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Version française abrégée

Dans cette note, nous étudions le comportement des solutions du problème de Neumann pour l'équation de Laplace (1) dans un domaine mince à frontière fortement oscillante. Nous considérons comme principale nouveauté, en nous éloignant du cas purement périodique, une famille de domaines minces $R^\epsilon \subset \mathbb{R}^2$, $\epsilon > 0$, où l'amplitude et la fréquence des oscillations varient, voir (2). La frontière oscillante est donnée par une fonction $G(\cdot,\cdot)$ de telle sorte que, pour chaque $x \in (0,1)$, la

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fonction $G(x,\cdot)$ est $l(x)$-périodique, où la fonction $l(x)$ est régulière. Nous nous intéressons à analyser l'influence de la géométrie localement périodique des domaines sur le problème limite.

Pour obtenir le problème limite homogénéisé, nous étendons la méthode de l'éclatement, introduite par D. Cioranescu, A. Damlamian et G. Griso [13], aux problèmes définis dans les domaines minces à frontière localement périodique. Pour cela, nous définissons un opérateur d'éclatement (Définition 2.1), qui vérifie les propriétés principales données en [14].

Étant donné qu'il n'existe aucune cellule fixe représentative qui puisse capturer la géométrie du domaine, le point clé pour obtenir l'opérateur d'éclatement adéquat est la construction du partage (3) de l'intervalle $[0,1]$, qui permet de diviser le domaine, montrant ainsi, d'une certaine manière, sa structure localement périodique. Grâce au bon choix des points du partage, nous obtenons le critère d'éclatement pour les intégrales (4) et la convergence des supports des fonctions images par l'opérateur d'éclatement au domaine fixe espéré (voir Proposition 2.2).

Le résultat principal de notre travail, énoncé dans le Théorème 2.2, établit les convergences nécessaires pour obtenir le problème limite (8) en choisissant des fonctions tests adéquates à la formulation variationnelle de (1).

Dans la Section 4, nous donnons un résultat de correcteurs basé sur la convergence de l'énergie et les propriétés de l'opérateur adjoint de $\mathcal{T}_\epsilon$.

Les démonstrations détaillées seront présentées dans le travail [6].

1. Introduction

This Note is devoted to study the behavior of solutions of the Neumann problem for the Laplace operator:

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = f^\epsilon & \text{in } R^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon, \end{cases} \tag{1}$$

where $f^\epsilon \in L^2(R^\epsilon)$, $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and R^ϵ is a two-dimensional thin domain with a highly oscillatory behavior at the boundary, which is given by

$$R^\epsilon = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon G(x, y/\epsilon)\}, \tag{2}$$

where $G(\cdot,\cdot)$ is a smooth function satisfying $0 < G_0 \leqslant G(\cdot,\cdot) \leqslant G_1$ for some fixed positive constants G_0, G_1 and $G(x,\cdot)$ is $l(x)$-periodic. Notice that although for fixed x the function $y \to G(x,y)$ has a periodic structure, both the amplitude and the period vary as we vary x. In this respect, we are deviating from the purely periodic case, which is the most common setting in homogenization theory and we are interested in understanding the influence of both the varying amplitude and period of the function G in the limiting homogenized problem.

The behavior of the solutions for elliptic partial differential equations on thin domains has been studied in different works in the literature. We would like to mention some of them. The case where the thin domain does not present oscillations was investigated in [18,21], the purely periodic case was addressed in [20] using standard techniques in homogenization theory, as developed in [7,12,22], and the locally periodic case with constant period was analyzed in [3]. Other related works are [5,4].

Let us also point out that there are many papers addressing the problem of studying the effect of rough boundaries on the behavior of the solution of partial differential equations. Among others, we can mention [1,8,17,19] in the context of the wall laws for fluid flows and [10,11], where complete asymptotic expansions of the solutions were studied.

Actually, our work differs from the previous articles, since we consider thin domains where the amplitude and frequency of the oscillations depend on x. We will get the associated homogenized limit problem, together with a corrector result by extending the periodic unfolding method, introduced by D. Cioranescu, A. Damlamian, and G. Griso in [13,14], to problems defined on thin domains with locally periodic oscillatory boundary. This extension is carried over the case of thin domain, but the ideas and techniques are applied to other cases, like perforated or reticulated domains (not necessarily thin). We refer to [15,9] for other applications of the unfolding method.

2. The unfolding operator in a thin domain with locally periodic oscillatory boundary

In this section, we define an unfolding operator for thin domains with locally periodic oscillatory boundary. A delicate point is to get a suitable partition of the limit segment $I = (0,1)$, which will match the oscillatory behavior of the thin domain (2). Observe that it is not possible to choose a basic reference cell Y^* such that the domain R^ϵ is obtained by the union of the translates of ϵY^*. To overcome this difficulty, we will define a partition of the interval $(0,1)$ through the function period $l(\cdot)$. To do that, we need to assume that:

(H) $l(\cdot)$ is a smooth function verifying $0 < l_0 \leqslant l(x) < l_1$ for two constants l_0, l_1 and $l'(x) < \frac{l(x)}{x}$ for all $x \in I$.

Then, for every ϵ fixed, we consider the points $x_k^\epsilon \in (0,1)$ such that $\frac{x_k^\epsilon}{l(x_k^\epsilon)} = k\epsilon$, for some $k \in \mathbb{N}$ and we get the following partition of the interval $[0,1]$:

$$x_0^\epsilon = 0 < x_1^\epsilon = \epsilon l(x_1^\epsilon) < x_2^\epsilon = 2\epsilon l(x_2^\epsilon) < \cdots < x_{N^\epsilon}^\epsilon = N^\epsilon \epsilon l(x_{N^\epsilon}^\epsilon) < x_{N^\epsilon+1}^\epsilon = 1. \tag{3}$$

Remark 1. The partition given is correctly defined by assumption **(H)**. This hypothesis guarantees that the function $\frac{x}{l(x)}$ is strictly increasing. Moreover, notice that the partition is not equally spaced but somehow reproduces the locally periodic structure of the thin domain.

Hence, for every $x \in [0, 1]$, there exists a unique element of the partition, x_k^ϵ, such that $x \in (x_k^\epsilon, x_{k+1}^\epsilon)$. Using similar notations to [14], we denote this x_k^ϵ by $[x]_\epsilon$. In addition, set:

$$\Gamma_{[x]_\epsilon} = \frac{x_{k+1}^\epsilon - x_k^\epsilon}{l(x_k^\epsilon)} \quad \forall x \in (x_k^\epsilon, x_{k+1}^\epsilon).$$

Now, if $Y^* = (0, l_1) \times (0, G_1)$ and $\widetilde{\varphi}$ denotes the standard extension by zero for $\varphi \in L^2(R^\epsilon)$ we may define the following operator:

Definition 2.1. We define the unfolding operator $\mathcal{T}_\epsilon : L^2(R^\epsilon) \to L^2((0, 1) \times Y^*)$ as

$$L^2(R^\epsilon) \ni \varphi \to \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2) = \begin{cases} \widetilde{\varphi}([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2) & \text{for } y_1 \in (0, l([x]_\epsilon)), \\ 0 & \text{for } y_1 \in (l([x]_\epsilon), l_1). \end{cases}$$

Obviously, $\mathcal{T}_\epsilon$ is a linear operator and $\mathcal{T}_\epsilon(\varphi\psi) = \mathcal{T}_\epsilon(\varphi)\mathcal{T}_\epsilon(\psi)$, $\forall \varphi, \psi \in L^2(R^\epsilon)$. Furthermore, it satisfies the following unfolding criterion for integrals, which will allow us to transform integrals over R^ϵ into integrals over $(0, 1) \times Y^*$.

Proposition 2.1. *Unfolding criterion for integrals (u.c.i.):*

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)\, dx\, dy_1\, dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x, y)\, dx\, dy, \quad \forall \varphi \in L^2(R^\epsilon),$$

$$\frac{1}{\epsilon}\int_{R^\epsilon} l([x]_\epsilon)\varphi(x, y)\, dx\, dy = \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)\, dx\, dy_1\, dy_2 \quad \forall \varphi \in L^2(R^\epsilon). \tag{4}$$

Remark 2. Since the domains R^ϵ collapse themselves to the unit interval $(0, 1)$ when $\epsilon \to 0$, we consider the following norms:

$$|||\varphi|||_{L^2(R^\epsilon)} = \epsilon^{-1/2}\|\varphi\|_{L^2(R^\epsilon)}, \quad \forall \varphi \in L^2(R^\epsilon), \qquad |||\varphi|||_{H^1(R^\epsilon)} = \epsilon^{-1/2}\|\varphi\|_{H^1(R^\epsilon)}, \quad \forall \varphi \in H^1(R^\epsilon).$$

Then, as a consequence of (4), the unfolding operator is continuous and the following relationships exist between the norms:

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^2((0,1)\times Y^*)} \leqslant \sqrt{l_1}|||\varphi|||_{L^2(R^\epsilon)}, \quad |||\varphi|||_{L^2(R^\epsilon)} \leqslant \frac{1}{\sqrt{l_0}}\|\mathcal{T}_\epsilon(\varphi)\|_{L^2((0,1)\times Y^*)}. \tag{5}$$

Now, we investigate the convergence properties related with the unfolding operator when $\epsilon \to 0$. Notice that one of the main differences of this unfolding operator in relation to the more standard periodic unfolding operator is that the support of the functions $\mathcal{T}_\epsilon(\varphi)$ depends on ϵ. However, with the choice of the partition points $\{x_k^\epsilon\}_k$, we will see that the support converges in a certain sense to the fixed domain $W = \{(x, y_1, y_2) \in \mathbb{R}^3\colon\ x \in (0, 1),\ (y_1, y_2) \in Y^*(x)\}$. As a matter of fact, if χ^ϵ denotes the characteristic function of R^ϵ and χ the characteristic function of W, we obtain the following proposition.

Proposition 2.2. *With the notations above we have* $\mathcal{T}_\epsilon\chi^\epsilon \to \chi\ s - L^2((0, 1) \times Y^*)$.

Proof. Taking into account the regularity properties of $G(\cdot,\cdot)$ and $l(\cdot)$, we only need to prove the following convergences for every $x \in (0, 1)$ and $y_1 \in (0, l(x))$:

$$[x]_\epsilon \xrightarrow{\epsilon\to 0} x, \qquad [x]_\epsilon + \Gamma_{[x]_\epsilon} y_1 \xrightarrow{\epsilon\to 0} x. \tag{6}$$

$$\frac{1}{\epsilon}\left([x]_\epsilon - l([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1)\frac{[x]_\epsilon}{l([x]_\epsilon)} + \Gamma_{[x]_\epsilon} y_1\right) \xrightarrow{\epsilon\to 0} y_1. \tag{7}$$

Both convergences from (6) are immediate from the definition of $[x]_\epsilon$ and $\Gamma_{[x]_\epsilon}$. Due to the assumption **(H)** and the good selection of the points of the partition (3) we get the limit

$$\frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 \overset{\epsilon\to 0}{\longrightarrow} \left(1-\frac{x}{l(x)}l'(x)\right)^{-1} y_1.$$

Then, using this convergence is easy to obtain (7). □

We also can show,

Proposition 2.3. *Let $\varphi \in L^2(0,1)$. Then, considering φ as a function defined in R^ϵ by $\varphi(x,y)=\varphi(x)$, we have:*

$$\mathcal{T}_\epsilon(\varphi) \longrightarrow \varphi\chi \quad s-L^2\big((0,1)\times Y^*\big).$$

Proposition 2.4. *Let $\psi \in C^\infty(\overline{W})$ be an $l(x)$-periodic function in the second variable for every fixed x. We define $\psi^\epsilon \in H^1(R^\epsilon)$ by*

$$\psi^\epsilon(x,y)=\psi\left(x,\frac{x}{\epsilon},\frac{y}{\epsilon}\right) \quad \forall (x,y)\in R^\epsilon.$$

Then

$$\mathcal{T}_\epsilon\big(\psi^\epsilon\big) \longrightarrow \psi\chi \quad s-L^2\big((0,1)\times Y^*\big).$$

For simplicity, we denote the space of functions $\varphi \in L^2(W)$ such that $\frac{\partial\varphi}{\partial y_1}$, $\frac{\partial\varphi}{\partial y_2}$ belong to $L^2(W)$ and $\varphi(x,\cdot,\cdot)$ is $l(x)$-periodic in the first variable by $L^2((0,1);{H^1}_\#(Y^*(x)))$.

Now, we can state the main result.

Theorem 2.2. *Let $\varphi^\epsilon \in H^1(R^\epsilon)$ for every ϵ, with $|||\varphi|||_{H^1(R^\epsilon)}$ uniformly bounded. Then,*

(i) *There exists a function φ in $H^1(0,1)$ such that, up to subsequences:*

$$\mathcal{T}_\epsilon\big(\varphi^\epsilon\big) \rightharpoonup \hat{\varphi}=\varphi\chi \quad w-L^2\big((0,1)\times Y^*\big).$$

(ii) *There exists a function φ_1 in $L^2((0,1);{H^1}_\#(Y^*(x)))$ such that, up to subsequences:*

$$\mathcal{T}_\epsilon\left(\frac{\partial\varphi^\epsilon}{\partial x}\right) \rightharpoonup \xi_0(x,y_1,y_2)=\begin{cases}\frac{\partial\varphi}{\partial x}(x)+l(x)\frac{\partial\varphi_1}{\partial y_1}(x,y_1,y_2) & \text{for } (x,y_1,y_2)\in W\\ 0 & \text{for } (x,y_1,y_2)\in(0,1)\times Y^*\setminus W.\end{cases}$$

$$\mathcal{T}_\epsilon\left(\frac{\partial\varphi^\epsilon}{\partial y}\right) \rightharpoonup \xi_1(x,y_1,y_2)=\begin{cases}l(x)\frac{\partial\varphi_1}{\partial y_2}(x,y_1,y_2) & \text{for } (x,y_1,y_2)\in W\\ 0 & \text{for } (x,y_1,y_2)\in(0,1)\times Y^*\setminus W.\end{cases}$$

Proof. We will give some ideas on how the proof of this result can be accomplished.

i) Since the norm $|||\varphi^\epsilon|||_{L^2(R^\epsilon)}$ is uniformly bounded, using (5) we can extract a subsequence of $\mathcal{T}_\epsilon(\varphi^\epsilon)$, still denoted by $\mathcal{T}_\epsilon(\varphi^\epsilon)$ such that $\mathcal{T}_\epsilon(\varphi^\epsilon) \rightharpoonup \hat{\varphi}$ $w-L^2((0,1)\times Y^*)$. Checking that $\hat{\varphi}$ is zero outside W is easy from Definition 2.1 and Proposition 2.2.

In order to prove that $\hat{\varphi}$ does not depend on (y_1,y_2) in W, we consider the test function $\Psi^\epsilon \equiv (\psi_1,\psi_2)(x,\frac{x}{\epsilon},\frac{y}{\epsilon}) \in [\mathcal{D}(R^\epsilon)]^2$ where $\Psi=(\psi_1,\psi_2)$ is a function in $[\mathcal{D}(W)]^2$. Integrating by parts, we obtain:

$$\int_{R^\epsilon} \nabla\varphi^\epsilon(x,y)\cdot\Psi\left(x,\frac{x}{\epsilon},\frac{y}{\epsilon}\right)\mathrm{d}x\,\mathrm{d}y=-\int_{R^\epsilon}\varphi^\epsilon(x,y)\left(\frac{\partial\psi_1}{\partial x}\left(x,\frac{x}{\epsilon},\frac{y}{\epsilon}\right)+\frac{1}{\epsilon}\,\mathrm{div}_{y_1y_2}\Psi\left(x,\frac{x}{\epsilon},\frac{y}{\epsilon}\right)\right)\mathrm{d}x\,\mathrm{d}y.$$

Then, by the criterion for integrals and passing to the limit, we get:

$$0=-\int_W \frac{1}{l(x)}\hat{\varphi}(x,y_1,y_2)\,\mathrm{div}_{y_1y_2}\Psi(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2 \quad \forall\Psi\in\big[\mathcal{D}(W)\big]^2.$$

This implies that there exists a function $\varphi\in L^2(0,1)$ such that: $\hat{\varphi}=\varphi\chi$. Finally, to show that $\varphi\in H^1(0,1)$, we use similar arguments as those in [2].

ii) In order to find the precise expression of ξ_0 and ξ_1, we consider a function $\Psi\equiv(\psi_1,\psi_2)\in[L^2((0,1);{H^1}_\#(Y^*(x)))\cap C^\infty(\overline{W})]^2$ verifying,

$$\mathrm{div}_{y_1y_2}\Psi=0$$
$$\Psi(x,y_1,y_2)\cdot n_{(y_1,y_2)}(x)=0 \quad \text{on } \partial_{\inf}Y^*(x)\cup\partial_{\sup}Y^*(x),$$

where $n_{(y_1,y_2)}(x)$ is the outward normal to $\partial Y^*(x)$ for every $x \in (0,1)$. From Proposition 2.4, we can define $\Psi^\epsilon \equiv (\psi_1{}^\epsilon, \psi_2{}^\epsilon) \in [H^1(R^\epsilon)]^2$, where

$$\psi_i{}^\epsilon(x,y) = \psi_i\left(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\right) \quad \forall (x,y) \in R^\epsilon,\ i = 1,2.$$

Then, by integrating by parts, we have:

$$\int_{R^\epsilon} \left[\nabla\varphi^\epsilon(x,y) - \nabla\varphi(x)\right] \cdot \Psi^\epsilon(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\int_{R^\epsilon} \left[\varphi^\epsilon(x,y) - \varphi(x)\right] \frac{\partial\psi_1}{\partial x}\left(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\right) \mathrm{d}x\,\mathrm{d}y.$$

Applying the unfolding criterion for integrals and passing to the limit in both terms with the help of Proposition 2.2, Proposition 2.4 and the convergence in i), we get:

$$\int_W \left(\frac{1}{l(x)}\left[\xi_0(x,y_1,y_2) - \frac{\partial\varphi}{\partial x}(x)\right], \frac{1}{l(x)}\xi_1(x,y_1,y_2)\right) \cdot (\psi_1,\psi_2)(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2 = 0.$$

The Helmholtz decomposition, see [16], yields that the orthogonal of divergence-free functions is exactly the gradients. This completes the proof. □

Remark 3. The proof provided for Theorem 2.2 uses similar techniques as in [2]. It is also possible to obtain the same result by adapting the scale-splitting operators, $\mathcal{Q}_\epsilon$ and $\mathcal{R}_\epsilon$, introduced in [13], to the new situation presented in this paper. However, the calculations using these operators are a little bit more elaborate.

3. Homogenization of the Neumann problem

In this section, we apply the unfolding operator defined in Definition 2.1 to obtain the homogenized limit equation for the problem (1) presented in the Introduction.

Theorem 3.1. *Let u^ϵ be the solution of problem (1). Assume to simplify that the nonhomogeneous term f^ϵ is given by $f^\epsilon(x,y) = f(x) \forall x \in (0,1)$, with $f \in L^2(0,1)$. Then, there exists $u \in H^1(0,1)$ such that*

$$\mathcal{T}_\epsilon(u^\epsilon) \rightharpoonup u\chi \quad \text{weakly in } L^2\big((0,1) \times Y^*\big),$$

and it is the unique weak solution of the following Neumann problem

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)u_x\big)_x + u = f, & x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{8}$$

where

$$r(x) = \frac{1}{l(x)} \int_{Y^*(x)} \left\{1 - \frac{\partial X(x)}{\partial y_1}(y_1,y_2)\right\} \mathrm{d}y_1\,\mathrm{d}y_2$$

and $X(x)$ is the unique solution (up to constants) which is $l(x)$-periodic in the first variable of the problem:

$$\begin{cases} -\Delta X(x) = 0 & \text{in } Y^*(x) \\ \dfrac{\partial X(x)}{\partial N} = 0 & \text{on } B_2(x) \\ \dfrac{\partial X(x)}{\partial N} = N_1(x) & \text{on } B_1(x) \end{cases}$$

in the representative cell $Y^*(x) = \{(y_1,y_2) \in \mathbb{R}^2\colon\ 0 < y_1 < l(x), 0 < y_2 < G(x,y_1)\}$. Also, $B_1(x)$ is the upper boundary and $B_2(x)$ is the lower boundary of $\partial Y^*(x)$ for all $x \in I$.

Proof. We use the unfolding operator introduced above. We start by establishing a priori estimates of u^ϵ. In fact, taking $\varphi = u^\epsilon$ in the variational formulation of (1), we obtain that there exists $C > 0$, independent of ϵ, such that $|||u^\epsilon|||_{H^1(R^\epsilon)} \leqslant C$. Therefore, the compactness Theorem 2.2 implies that there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1{}_\#(Y^*(x)))$ such that

$$\mathcal{T}_\epsilon(u^\epsilon) \rightharpoonup \hat{u} = u\chi \quad \text{weakly in } L^2\big((0,1)\times Y^*\big),$$
$$\mathcal{T}_\epsilon\left(\frac{\partial u^\epsilon}{\partial x}\right) \rightharpoonup \xi_0(x,y_1,y_2) = \frac{\partial u}{\partial x}(x) + l(x)\frac{\partial u_1}{\partial y_1}(x,y_1,y_2) \quad \text{weakly in } L^2(W),$$
$$\mathcal{T}_\epsilon\left(\frac{\partial u^\epsilon}{\partial y}\right) \rightharpoonup \xi_1(x,y_1,y_2) = l(x)\frac{\partial u_1}{\partial y_2}(x,y_1,y_2) \quad \text{weakly in } L^2(W).$$

We apply the unfolding operator to the variational formulation of (1), taking as test function the function $\phi(x) + \epsilon\psi(x, \frac{x}{\epsilon}, \frac{y}{\epsilon})$, where $\phi \in H^1(0,1)$ and $\psi \in L^2((0,1); H^1{}_\#(Y^*(x))) \cap C^\infty(\overline{W})$. Then, passing to the limit, we obtain:

$$\begin{cases} \forall\phi \in H^1(0,1), \quad \forall\psi \in L^2\big((0,1); H^1{}_\#\big(Y^*(x)\big)\big) \\ \displaystyle\int_W \bigg\{\bigg(\frac{1}{l(x)}\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1,y_2)\bigg)\bigg(\frac{\partial \phi}{\partial x}(x) + \frac{\partial \psi}{\partial y_1}(x,y_1,y_2)\bigg) \\ \quad + \displaystyle\frac{\partial u_1}{\partial y_2}(x,y_1,y_2)\frac{\partial \psi}{\partial y_2}(x,y_1,y_2) + \frac{u(x)\phi(x)}{l(x)}\bigg\}\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2 = \displaystyle\int_W \frac{f(x)\phi(x)}{l(x)}\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2. \end{cases} \tag{9}$$

Observe that we can consider $\psi \in L^2((0,1); H^1{}_\#(Y^*(x)))$ by density. By taking $\phi = 0$ in (9), it is easy to see that $u_1(x,y_1,y_2) = -X(x)(y_1,y_2)\frac{1}{l(x)}\frac{\partial u}{\partial x}(x)$ for $(x,y_1,y_2) \in W$. Finally, considering $\psi = 0$ and replacing u_1 by this value in Eq. (9) yields the weak formulation of (8).

The uniqueness and existence of the weak solution to problem (8) is an immediate consequence of the Lax–Milgram theorem. □

Remark 4. In case the period is constant, $l(x) \equiv L$, we obtain the same homogenized limit problem as in [3].

4. Convergence of the energy and correctors

We sketch in this section some results on the correctors. We start defining the operator $\mathcal{U}_\epsilon$, adapted to the situation of locally periodic thin domains.

Definition 4.1. For $\varphi \in L^2((0,1)\times Y^*)$, we set:

$$\mathcal{U}_\epsilon(\varphi)(x,y) = \int_{(0,l([x]_\epsilon))} \frac{1}{l([x]_\epsilon)}\varphi\left([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{x-[x]_\epsilon}{\Gamma_{[x]_\epsilon}}, \frac{y}{\epsilon}\right)\mathrm{d}y_1, \quad \forall (x,y) \in R^\epsilon.$$

Some properties of this operator are the following.

Proposition 4.1. *$\mathcal{U}_\epsilon$ has the following properties:*

(i) *For every $\varphi \in L^2((0,1)\times Y^*)$ one has $|||\mathcal{U}_\epsilon(\varphi)|||_{L^2(R^\epsilon)} \leqslant C\|\varphi\|_{L^2((0,1)\times Y^*)}$.*
(ii) *$\mathcal{U}_\epsilon$ is the left inverse of $\mathcal{T}_\epsilon$, that is $(\mathcal{U}_\epsilon \circ \mathcal{T}_\epsilon)(\phi) = \phi$ for every $\phi \in L^2(R^\epsilon)$.*
(iii) *Let $\varphi \in L^2(0,1)$. Then, $|||\mathcal{U}_\epsilon(\varphi) - \varphi|||_{L^2(R^\epsilon)} \to 0$ when $\epsilon \to 0$.*
(iv) *Suppose that $\{\varphi^\epsilon\}$ is a sequence in $L^2(R^\epsilon)$. If $\mathcal{T}_\epsilon(\varphi^\epsilon) \longrightarrow \varphi\ s - L^2((0,1)\times Y^*)$ then $|||\mathcal{U}_\epsilon(\varphi) - \varphi^\epsilon|||_{L^2(R^\epsilon)} \to 0$.*

Proposition 4.2. *Assume that hypotheses of Theorem 3.1 are satisfied. Then*

$$\mathcal{T}_\epsilon\big(\nabla_{xy}u^\epsilon\big) \longrightarrow \big(\nabla_{xy}u + l(x)\nabla_{y_1y_2}u_1\big)\chi \quad s - L^2\big((0,1)\times Y^*\big).$$

Using the operator $\mathcal{U}_\epsilon$ and with some additional regularity requirements on the functions G and l, we obtain the corrector result:

Theorem 4.2. *Assume hypotheses of Theorem 3.1 hold. Then,*

(i) $\lim_{\epsilon\to 0} |||u^\epsilon - u|||_{L^2(R^\epsilon)} = 0$.
(ii) $\lim_{\epsilon\to 0} |||\nabla u^\epsilon - \nabla u - l(x)\mathcal{U}_\epsilon(\nabla_{y_1y_2}u_1)|||_{[L^2(R^\epsilon)]^2} = 0$.
(iii) $\lim_{\epsilon\to 0} |||u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X(x, x/\epsilon, y/\epsilon)|||_{H^1(R^\epsilon)} = 0$.

Proof. Convergence (i) can be proved using extension operators. Convergence (ii) follows from the convergence of the energy (Proposition 4.2) and Proposition 4.1. Finally, convergence (iii) is a consequence of the two above. □

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