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A remark on the approximation of plurisubharmonic functions

Une remarque sur l'approximation des fonctions pluri-sous-harmoniques

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ABSTRACT

We show by an example that the Demailly approximation sequence of a plurisubharmonic function, constructed via Bergman kernels, is not a decreasing sequence in general.

RÉSUMÉ

Nous montrons par un exemple que le résultat de Demailly relatif à l'approximation d'une fonction pluri-sous-harmonique via les noyaux de Bergman ne produit pas en général une suite décroissante.

1. Introduction

It is a fundamental theorem of Demailly [2] that given an arbitrary plurisubharmonic function φ on a domain, there always exists a sequence $\{\varphi_m\}$ of plurisubharmonic functions with analytic singularities converging to φ. Moreover, the approximating function φ_m is given in a very natural form: $\varphi_m = \frac{1}{2m} \log \sum |\sigma_l|^2$, where (σ_l) is an orthonormal basis of the Hilbert space of holomorphic functions that are square integrable with respect to the weight $e^{-2m\varphi}$. (See [2, Proposition 3.1], [3] and also the exposition in [1].)

It was further proved that the subsequence $\{\varphi_{2^k}\}$ is decreasing (after adding suitable constants) in [4, Step 3, Proof of Theorem 2.3] using a subadditivity property of the sequence φ_m's. It remained a natural question, raised explicitly in [1, p. 134], to ask whether the entire sequence $\{\varphi_m\}$ is decreasing or not, even up to the *equivalence of singularities* (see the beginning of the next section for its definition). In this note, we show by an example that the Demailly approximation sequence of a plurisubharmonic function is not necessarily decreasing, thus answering the above question negatively. The example φ is given as a plurisubharmonic function with analytic singularities, for which we can compute the multiplier ideal sheaf of each $m\varphi$ and determine the singularities of φ_m using a finite number of local generators of $\mathcal{J}(m\varphi)$.

2. The example

We recall the following definitions from [3]. For two singular weights $h_i = e^{-\varphi_i}$ $(i = 1, 2)$, we say h_1 is **less singular** than h_2 and write $h_1 \preccurlyeq h_2$ if the quotient $\frac{h_1}{h_2}$ is locally bounded above. In this case, we also write $\varphi_2 \preccurlyeq \varphi_1$. If $h_1 \preccurlyeq h_2$ and $h_2 \preccurlyeq h_1$, we say h_1 and h_2 **have equivalent singularities** and write $h_1 \sim h_2$ and $\varphi_1 \sim \varphi_2$.

Let X be a complex manifold and $D \subset X$ be an irreducible hypersurface. For a local defining function f of D, we associate the plurisubharmonic function $\varphi_D := \log |f|^2$ (at least locally). More generally, let us denote by φ_D the plurisubharmonic

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function corresponding to an effective divisor $D=\sum_{i=1}^k a_iD_i$, that is, $\varphi_D=\sum_{i=1}^k \log|f_i|^{2a_i}$. Here f_i is a local equation of D_i and as is well known, φ_D makes sense globally as local weight functions of a singular Hermitian metric [3] (of $\mathcal{O}(D)$ if it is a line bundle).

We first give a trivial example of such φ_D when $D\subset X$ is a smooth hypersurface. In this case, $\mathcal{J}(m\varphi_D)=\mathcal{O}(-mD)$, which is locally generated by one generator f^m where $D=(f=0)$. It follows that, in this case, the Demailly approximation $\{\varphi_m\}$ is a constant sequence, at least up to equivalence of singularities.

For a simple example with nonconstant $\{\varphi_m\}$, let $X=\mathbf{C}^2$ (with coordinates x,y) and $\varphi=\log(|x|^2+|y|^2)$. Since $\mathcal{J}(m\varphi)=\mathfrak{m}^{m-1}$ (where $\mathfrak{m}\subset\mathcal{O}_X$ is the maximal ideal of the point $(0,0)$), it follows from considering the generators of the ideal $\mathfrak{m}^{m-1}$ that $\varphi_m=\frac{1}{m}\log(\sum_{i=0}^{m-1}|x^iy^{m-1-i}|^2)\sim\frac{m-1}{m}\log(|x|^2+|y|^2)$, which is strictly decreasing in terms of equivalence of singularities.

In order to construct our example, let us take $X=\mathbf{C}^2$ (with coordinates x,y) and the effective divisor $D=\sum_{i=1}^3 a_iD_i$ ($a_i\geqslant 0$), where $D_1=(x=0)$, $D_2=(y=0)$, $D_3=(z:=x+y=0)$. This is the simplest example of a non-SNC divisor. One blowup of the origin $\pi: X'\to X$ gives the log-resolution of the pair (X,D) and we can compute the multiplier ideal sheaves.

Let $H_i\subset X'$ be the proper transform of D_i and E the exceptional divisor of π. Then the multiplier ideal sheaves are given by [3,5]:

$$\mathcal{J}(m\varphi_D)=\mathcal{J}(mD)=\pi_*\mathcal{O}_{X'}\big(K_{X'}-\pi^*(K_X)-\lfloor\pi^*(mD)\rfloor\big)=\pi_*\mathcal{O}_{X'}\left(-\sum_{i=1}^3\lfloor ma_i\rfloor H_i-(2m-1)E\right).$$

Now let $\Omega\subset\mathbf{C}^2$ be a connected Stein neighborhood of the origin and consider the Demailly approximation sequence φ_m on Ω [2, Proposition 3.1] given by the Bergman kernels corresponding to the weight function $\mathrm{e}^{-m\varphi_D}$. From now on, we identify a plurisubharmonic function with its equivalence class in terms of singularities as in [3, Definition 6.3]. If the sequence $\{\varphi_m\}$ is genuinely decreasing (up to some constants), then of course it is also decreasing in terms of equivalence of singularities.

Since φ_m is equivalent to a plurisubharmonic function with analytic singularities given by a finite number of generators of the multiplier ideal sheaf $\mathcal{J}(m\varphi_D)$ in a relatively compact Stein open subset of Ω, we can use it to show the following.

Theorem 2.1. *Let $\{\varphi_m\}$ be the Demailly approximation sequence for the plurisubharmonic function $\varphi=\varphi_D$ where D is as in the above with $a_1=a_2=a_3=\frac{2}{3}$. Then the sequence is not decreasing in the sense that we cannot choose a sequence of constants C_m such that φ_m+C_m is a decreasing sequence of $\mathbf{R}\cup\{-\infty\}$-valued functions.*

The coefficients $a_1=a_2=a_3=\frac{2}{3}$ are chosen here only for the simplicity of computation. Apparently, the behavior of the approximation sequence as in this theorem are expected to be very common for other psh functions.

Proof of Theorem 2.1. Let $\varphi=\varphi_D$. From now on, functions have the domain as a unit ball around the origin. For $m\geqslant 1$, the psh function $2\varphi_m$ is equivalent to $\frac{1}{m}\log(\sum|f_i|^2)$, where the f_i's are the finite number of generators for $\mathcal{J}(m\varphi)$. As for the multiplier ideal sheaf $\mathcal{J}(2\varphi)=\pi_*\mathcal{O}_{X'}(-\sum H_i'-3E)$, we see that it can be generated (over $\mathcal{O}_X$) by $f_1=xyz$. Hence, for $m=2$, we have $2\varphi_2\sim\log|xyz|$. Similarly, for $\mathcal{J}(3\varphi)=\pi_*\mathcal{O}_{X'}(-\sum 2H_i'-5E)$, the generator can be chosen as $(xyz)^2$, thus $2\varphi_3\sim\log|xyz|^{\frac{4}{3}}$.

One then immediately sees that the sequence of singularities is not decreasing: $\varphi_5\not\preccurlyeq\varphi_3$ since $2\varphi_5\sim\log|xyz|^{\frac{6}{5}}$. In fact, we see that $\varphi_5\succeq\varphi_3$ holds in this case. Also one checks that $\varphi_4\not\preccurlyeq\varphi_3$: ($\mathcal{J}(4\varphi)$ has generators $(xyz)^2x$, $(xyz)^2y$, $(xyz)^2z$, the last being redundant, but included for convenience):

$$\frac{\mathrm{e}^{2\varphi_4}}{\mathrm{e}^{2\varphi_3}}\sim\frac{|xyz|(|x|^2+|y|^2+|z|^2)^{\frac{1}{4}}}{|xyz|^{\frac{4}{3}}}$$

is not locally bounded above, considering along $x=y$. This completes the proof of Theorem 2.1. □

Furthermore, similar computations yield that there is an infinite number of instances where the decreasing property fails to hold, considering m modulo 3 for the coefficient $\lfloor\frac{2m}{3}\rfloor$. We have:

$$2\varphi_{3k}\sim\log|xyz|^{2\frac{2k}{3k}}\preccurlyeq 2\varphi_{3k+2}\sim\log|xyz|^{2\frac{2k+1}{3k+2}}$$

generalizing $\varphi_5\succeq\varphi_3$. So we cannot truncate the sequence $\{\varphi_m\}$ to make it a decreasing one.

On the other hand, similar considerations can be used to check the decreasing property of the subsequence with indices of exponential growth [4, Step 3, Proof of Theorem 2.3] of our main example in Theorem 2.1 directly, without using subadditivity:

Proposition 2.2. $\varphi_{2^{k+1}}\preccurlyeq\varphi_{2^k}$ *for every* $k\geqslant 1$.

For the proof of this, we use the facts that $\lfloor 2^{2k}\frac{2}{3}\rfloor = 2\frac{2^{2k}-1}{3}$, $\lfloor 2^{2k+1}\frac{2}{3}\rfloor = 2\frac{2^{2k+2}-1}{3}$ and that $|x|^2+|y|^2+|z|^2 \geqslant 3|xyz|^{\frac{2}{3}}$. For the first few terms, we find

$$\frac{e^{2\varphi_8}}{e^{2\varphi_4}} \sim \frac{|xyz|^{\frac{10}{8}}}{|xyz|(|x|^2+|y|^2+|z|^2)^{\frac{1}{4}}} \preccurlyeq |xyz|^{\frac{1}{2}}$$

is locally bounded above.

$$\frac{e^{2\varphi_{16}}}{e^{2\varphi_8}} \sim \frac{|xyz|^{\frac{10}{8}}(|x|^2+|y|^2+|z|^2)^{\frac{1}{16}}}{|xyz|^{\frac{10}{8}}}$$

is locally bounded above thanks to the equality of the exponents of $|xyz|$ in the fraction. Then it is easy to check that these two patterns alternate and show $\varphi_{2^{k+1}} \preccurlyeq \varphi_{2^k}$ for every $k \geqslant 1$. This completes the proof of Proposition 2.2.

Finally, as Professor J.-P. Demailly kindly pointed out to us, we note that it might still be possible to show that there exists a (strictly) decreasing subsequence of the Demailly approximation with indices of linear growth, instead of exponential growth as above. In our main example, this is indeed the case since the subsequence $\varphi_{3k+2} \sim \log|xyz|^{\frac{2k+1}{3k+2}}$ is strictly decreasing and does converge to the original φ. The arguments used in this note might be possibly used for general φ with analytic singularities.

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