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Modeling of the nonlinear vibrations of a stiffened moderately thick plate

Modélisation des vibrations non linéaires d'une plaque raidie modérément épaisse

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ABSTRACT

We consider a multi-structure consisting of a plate reinforced by a thin stiffener on a portion of its boundary. The model we consider for this structure (viewed as a heterogeneous plate) is nonlinear and takes into account the transverse shear effects. Our aim is to model this junction and reproduce the effect of the thin stiffener by means of approximate boundary conditions on the junction region.

RÉSUMÉ

On s'intéresse à la modélisation d'une multi-structure composée d'une plaque élastique renforcée par un raidisseur mince et très rigide sur une partie de son bord. Les vibrations de cette structure (vue comme une plaque hétérogène) sont décrites par un modèle qui tient compte des effets du cisaillement transverse. Pour des raisons numériques, on modélise cette jonction et on propose un modèle approché qui ne fait pas intervenir le raidisseur, mais qui rend compte de son effet par de nouvelles conditions aux limites sur l'interface de jonction.

Version française abrégée

La simulation numérique du comportement des structures présentant des couches minces est difficile, car elle nécessite une discrétisation à l'échelle de l'épaisseur de la couche mince. Le maillage comporte alors beaucoup d'éléments et la taille du système linéaire qui en résulte explose, ce qui rend cette approche trés coûteuse. On cherche alors à remplacer la couche mince par des conditions aux limites qui rendent compte de son effet. Cette idée a fait l'objet de plusieurs études (voir [1,2,4–7]). Dans cette note, on se propose de modéliser le comportement d'une plaque renforcée par un raidisseur (couche mince très rigide) sur une partie de son bord. Dans [5], [6] et [7], des conditions aux limites approchées ont été justifiées pour des modèles bidimensionnels provenant de la théorie classique des plaques minces. Dans ce qui suit, on considère un modèle non linéaire qui relève de la théorie des plaques modérément épaisses et pour laquelle les effets de rotation et du cisaillement transverse ne peuvent être négligés.

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Soit $\delta > 0$ un paramètre destiné à tendre vers 0, décrivant l'épaisseur du raidisseur. La plaque occupe le domaine $\bar{\Omega}_+ = [0,1] \times [0,1]$, de bord $\partial\Omega_+ = \bar{\Sigma} \cup \bar{\Gamma}_+$, où $\Sigma =]0,1[\times \{0\}$. Elle est encastrée sur la partie Γ_+ et est renforcée par un raidisseur $\bar{\Omega}_-^\delta = [0,1] \times [-\delta,0]$ sur l'autre partie Σ. La frontière de celui-ci est notée $\partial\Omega_-^\delta = \bar{\Sigma}_-^\delta \cup \bar{\Sigma} \cup \bar{\Gamma}_-^\delta$, où $\Sigma_-^\delta =]0,1[\times \{-\delta\}$. On considère pour cette structure (vue comme une plaque rectangulaire occupant le domaine $\bar{\Omega}^\delta = [0,1] \times [-\delta,1]$), le modèle (1)–(5), où w, $u = (u_1, u_2)$ et ψ, ϕ désignent respectivement la flexion, le déplacement plan et les angles de rotation. Les coefficients ρ, E, μ et K désignent respectivement la densité de masse, le module de Young, le coefficient de Poisson et le module du cisaillement. On suppose que ρ, E et K sont indépendants de δ pour la plaque et varient en δ^{-1} pour le raidisseur. Une analyse asymptotique par rapport à δ conduit au modèle (10)–(12), qui est posé uniquement sur Ω_+ – excluant le domaine du raidisseur Ω_-^δ –, mais qui rend compte de son effet par les conditions aux limites (12). En effet, sa contribution se manifeste par les termes supplémentaires qui apparaissent à droite des conditions (12) et qui s'expriment en fonction de ses caractéristiques physiques. De nouvelles conditions initiales (9) apparaissent aussi sur l'interface Σ et traduisent aussi l'effet du raidisseur. Notons que ces conditions approchées ne sont pas classiques, car elles font intervenir des dérivées tangentielles et en temps du même ordre que celui de l'opérateur différentiel intérieur. Par ailleurs, le choix imposé sur l'ordre de grandeur des coefficients E, ρ, K, qui varient en δ^{-1} pour le raidisseur, est nécessaire pour avoir une contribution de celui-ci au problème limite. S'ils étaient du même ordre de grandeur que ceux de la plaque, le problème limite ne rendrait pas compte de l'effet du raidisseur.

1. Introduction

Numerical simulation of the behavior of structures involving thin layers is difficult since it requires the discretisation of the layer, which leads to very thin meshes and very expensive computations. An alternative approach consists in eliminating "the thin layer" geometrically and reproducing its effect by new boundary conditions on the junction region. So, we seek an equivalent model by reducing the layer to a boundary and by approximating its effect by conditions on this boundary. This idea has been used in many studies (see [1,2,4–7]). The present study deals with a plate reinforced by a thin stiffener on a portion of its boundary. The novelty in this note is to justify a new model that will be more accurate for "moderately" thick stiffened plates, whereas those proposed in [5–7] are just valid for thinner plates. Indeed, the effect of transverse shear deformation and the rotatory inertia become significant when the reinforced plate is relatively thick, and must be incorporated. After a suitable scaling, we examine the asymptotic behavior of the solution of the problem as the thickness δ of the thin body goes to zero. This analysis leads to a nonstandard boundary value problem in which the boundary conditions involve tangential and time derivatives of order equal to that of the interior differential operator.

We consider a bi-dimensional plate occupying the set $\bar{\Omega}_+ = [0,1] \times [0,1]$. The boundary of Ω_+ is denoted by $\partial\Omega_+ = \bar{\Sigma} \cup \bar{\Gamma}_+$, where $\Sigma =]0,1[\times \{0\}$. The plate is clamped on the portion Γ_+ of its boundary and is reinforced by a thin layer on the other part Σ. The thin body occupies the set $\bar{\Omega}_-^\delta = [0,1] \times [-\delta,0]$ of boundary $\partial\Omega_-^\delta = \bar{\Sigma}_-^\delta \cup \bar{\Sigma} \cup \bar{\Gamma}_-^\delta$, where $\Sigma_-^\delta =]0,1[\times \{-\delta\}$. These two elastic bodies form together an elastic multi-structure, viewed as a heterogeneous rectangular plate occupying the set $\bar{\Omega}^\delta = [0,1] \times [-\delta,1]$. Denoting by w and $u = (u_1, u_2)$, respectively, the vertical and horizontal displacement of the plate, ψ and ϕ the rotation angles of a filament of the latter, we consider the following nonlinear model (see [3]):

$$
\begin{aligned}
&\rho u_1'' - [\partial_x N_1 + \partial_y N_{12}] = 0, \qquad \rho u_2'' - [\partial_y N_2 + \partial_x N_{12}] = 0,\\
&\rho w'' - K\big[\partial_x(\partial_x w + \psi) + \partial_y(\partial_y w + \phi)\big] - \partial_x(N_1 \partial_x w + N_{12}\partial_y w) - \partial_y(N_2 \partial_y w + N_{12}\partial_x w) = 0,\\
&\rho\psi'' - D\left[\partial_x^2\psi + \frac{1-\mu}{2}\partial_y^2\psi + \frac{1+\mu}{2}\partial_{xy}^2\phi\right] + K[\psi + \partial_x w] = 0,\\
&\rho\phi'' - D\left[\partial_y^2\phi + \frac{1-\mu}{2}\partial_x^2\phi + \frac{1+\mu}{2}\partial_{xy}^2\psi\right] + K[\phi + \partial_y w] = 0 \quad \text{in } \Omega^\delta \times (0,T),
\end{aligned}
\tag{1}
$$

where $N_1 = \frac{E}{1-\mu^2}[\partial_x u_1 + \mu\partial_y u_2 + \frac{1}{2}(\partial_x w)^2 + \frac{\mu}{2}(\partial_y w)^2]$, $N_{12} = \frac{E}{2(1+\mu)}[\partial_y u_1 + \partial_x u_2 + \partial_x w \partial_y w]$ and $N_2 = \frac{E}{1-\mu^2}[\partial_y u_2 + \mu\partial_x u_1 + \frac{1}{2}(\partial_y w)^2 + \frac{\mu}{2}(\partial_x w)^2]$.

We define the Dirichlet (clamped) boundary conditions on the portion of the boundary $\Gamma_+ \cup \Gamma_-^\delta$:

$$
u_1 = 0, \qquad u_2 = 0, \qquad \psi = 0, \qquad \phi = 0, \qquad w = 0 \quad \text{on } \big(\Gamma_+ \cup \Gamma_-^\delta\big) \times (0,T). \tag{2}
$$

The boundary conditions on the boundary Σ_-^δ are given by:

$$
\begin{aligned}
&\nu_1 N_1 + \nu_2 N_{12} = 0, \qquad \nu_2 N_2 + \nu_1 N_{12} = 0,\\
&D\left[\nu_1\partial_x\psi + \mu\nu_1\partial_y\phi + \frac{1-\mu}{2}[\partial_y\psi + \partial_x\phi]\nu_2\right] = 0, \qquad D\left[\nu_2\partial_y\phi + \mu\nu_2\partial_x\psi + \frac{1-\mu}{2}[\partial_y\psi + \partial_x\phi]\nu_1\right] = 0,\\
&K[\partial_\nu w + \nu_1\psi + \nu_2\phi] + (\nu_1 N_1 + \nu_2 N_{12})\partial_x w + (\nu_2 N_2 + \nu_1 N_{12})\partial_y w = 0 \quad \text{on } \Sigma_-^\delta \times (0,T).
\end{aligned}
\tag{3}
$$

We define also the transmission conditions on Σ by:

$$\left[\!\left[D\left(\nu_1\partial_x\psi+\mu\nu_1\partial_y\phi+\frac{1-\mu}{2}[\partial_y\psi+\partial_x\phi]\nu_2\right)\right]\!\right]=\left[\!\left[D\left(\nu_2\partial_y\phi+\mu\nu_2\partial_x\psi+\frac{1-\mu}{2}[\partial_y\psi+\partial_x\phi]\nu_1\right)\right]\!\right]=0,$$
$$[\![K[\partial_\nu w+\nu_1\psi+\nu_2\phi]+(\nu_1N_1+\nu_2N_{12})\partial_x w+(\nu_2N_2+\nu_1N_{12})\partial_y w]\!]=0,$$
$$[\![\nu_1N_1+\nu_2N_{12}]\!]=[\![\nu_2N_2+\nu_1N_{12}]\!]=0; \qquad [\![u]\!]=0, \qquad [\![\psi]\!]=[\![\phi]\!]=[\![w]\!]=0 \quad \text{on } \Sigma\times(0,T). \tag{4}$$

We associate, with the system above, the initial conditions:

$$\psi(0)=\psi^*, \qquad \psi'(0)=\psi^{**}, \qquad w(0)=w^*, \qquad u(0)=u^*, \qquad u'(0)=u^{**},$$
$$w'(0)=w^{**}, \qquad \phi(0)=\phi^*, \qquad \phi'(0)=\phi^{**} \quad \text{in } \Omega^\delta, \tag{5}$$

where ψ^*, w^*, ϕ^*, u_1^*, $u_2^*\in H^1(\Omega^\delta)$ and ψ^{**}, w^{**}, ϕ^{**}, u_1^{**}, $u_2^{**}\in L^2(\Omega^\delta)$. $[\![\ ]\!]$ denotes the jump through Σ and $\nu=(\nu_1,\nu_2)$ is the inner unit normal to Σ. The constant ρ is the mass density and $D=\frac{E}{(1-\mu^2)}$ is the flexural rigidity of the plate of Young modulus E and a Poisson's ratio μ. The parameter K is the modulus of elasticity in shear. These coefficients are assumed to be of the form: $E=E_+$ in Ω_+ and $\delta^{-1}E_-$ in Ω_-^δ; $\mu=\mu_+$ in Ω_+ and μ_- in Ω_-^δ; $\rho=\rho_+$ in Ω_+ and $\delta^{-1}\rho_-$ in Ω_-^δ; $K=K_+$ in Ω_+ and $\delta^{-1}K_-$ in Ω_-^δ, where $E_\pm$, $\mu_\pm$, $\rho_\pm$, $K_\pm$ are independent of δ. The prime "′" stands for the time derivative.

1.1. Variational setting and scaling

Let $V(\Omega^\delta)=\{w\in H^1(\Omega^\delta);\ w|_{\Gamma_+\cup\Gamma_-^\delta}=0\}$ and $U(\Omega^\delta)=\{u\in H^1(\Omega^\delta)\times H^1(\Omega^\delta);\ u|_{\Gamma_+\cup\Gamma_-^\delta}=0\}$. The variational formulation of the problem (1)–(5) consists in finding $\psi,\phi,w\in L^\infty((0,T),V(\Omega^\delta))$; $\psi',\phi',w'\in L^\infty((0,T),L^2(\Omega^\delta))$, $u\in L^\infty(0,T;U(\Omega^\delta))$ and $u'\in L^\infty(0,T;(L^2(\Omega^\delta))^2)$ such that:

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}\big(w',u',\psi',\phi',\hat w,\hat u,\hat\psi,\hat\phi\big)+\mathcal{A}(\psi,\phi,\hat\psi,\hat\phi)+\mathcal{B}(w,\psi,\phi,\hat w,\hat\psi,\hat\phi)+\mathcal{N}(u,w,\hat u,\hat w)=0,\right. \tag{6}$$

for all $\{\hat\psi,\hat\phi,\hat w\}\in V(\Omega^\delta)$ and $\hat u\in U(\Omega^\delta)$, where:

- $\mathcal{A}(\psi,\phi,\hat\psi,\hat\phi)=\int_{\Omega^\delta}D(\partial_x\psi\partial_x\hat\psi+\partial_y\phi\partial_y\hat\phi+\mu\partial_x\psi\partial_y\hat\phi+\mu\partial_y\phi\partial_x\hat\psi+\frac{1-\mu}{2}(\partial_y\psi+\partial_x\phi)(\partial_y\hat\psi+\partial_x\hat\phi))\,\mathrm{d}\Omega^\delta$,
- $\mathcal{B}(w,\psi,\phi,\hat w,\hat\psi,\hat\phi)=\int_{\Omega^\delta}K[(\phi+\partial_y w)(\hat\phi+\partial_y\hat w)+(\psi+\partial_x w)(\hat\psi+\partial_x\hat w)]\,\mathrm{d}\Omega^\delta$,
- $\mathcal{C}(w',u',\psi,\phi',\hat w,\hat u,\hat\psi,\hat\phi)=\int_{\Omega^\delta}\rho(\psi'\hat\psi+\phi'\hat\phi+w'\hat w+u'\hat u)\,\mathrm{d}x\,\mathrm{d}y$,
- $\mathcal{N}(u,w,\hat u,\hat w)=\int_{\Omega^\delta}(N_1[\partial_x\hat u_1+\partial_x w\partial_x\hat w]+N_2[\partial_y\hat u_2+\partial_y w\partial_y\hat w]+N_{12}[\partial_y\hat u_1+\partial_x\hat u_2+\partial_x w\partial_y\hat w+\partial_y w\partial_x\hat w])\,\mathrm{d}x\,\mathrm{d}y$.

Existence and uniqueness of the solution of (6) may be shown by means of semi-group theory (see [8], for instance). We define the energy $E(t)$ by setting:

$$E(t)=\frac{1}{2}\big\{\rho\big[\|\psi'\|^2_{L^2(\Omega^\delta)}+\|w'\|^2_{L^2(\Omega^\delta)}+\|\phi'\|^2_{L^2(\Omega^\delta)}+\|u'\|^2_{(L^2(\Omega^\delta))^2}\big]+\mathcal{A}(\psi,\phi,\psi,\phi)$$
$$+\mathcal{N}(u,w,u,w)+\mathcal{B}(w,\psi,\phi,w,\psi,\phi)\big\}.$$

Before carrying out the asymptotic analysis of the problem (6) as $\delta\to 0$, we transform it into a problem posed on a domain $\Omega=]0,1[\times]-1,1[$ that does not depend on δ. So, we perform the change of scaling:

$$\left\{\begin{array}{l}\Omega_-=]0,1[\times]-1,0[\longrightarrow\Omega_-^\delta,\\(x,z)\longrightarrow(x,y)=(x,\delta z).\end{array}\right. \tag{7}$$

For a function ζ and a vector field $v=(v_1,v_2)$ defined on Ω_-^δ, we associate ζ^δ with v^δ, which are defined on Ω_- by: $\zeta^\delta(x,y)=\zeta(x,\delta z)$ and $v^\delta(x,y)=(v_1(x,\delta z),\ \delta v_2(x,\delta z))$. Furthermore, we have $\partial_y=\delta^{-1}\partial_z$. We denote by u_-^δ, w_-^δ, ψ_-^δ and ϕ_-^δ the functions obtained respectively from $u|_{\Omega_-^\delta}$, $w|_{\Omega_-^\delta}$, $\psi|_{\Omega_-^\delta}$ and $\phi|_{\Omega_-^\delta}$ through the scaling (7) and set $u^\delta=(u_+^\delta,u_-^\delta)$, $w^\delta=(w_+^\delta,w_-^\delta)$, $\psi^\delta=(\psi_+^\delta,\psi_-^\delta)$ and $\phi^\delta=(\phi_+^\delta,\phi_-^\delta)$, where $u_+^\delta=u|_{\Omega_+}$, $w_+^\delta=w|_{\Omega_+}$, $\psi_+^\delta=\psi|_{\Omega_+}$ and $\phi_+^\delta=\phi|_{\Omega_+}$. Thus, considering the variational problem (6), we split the forms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{N}$ into parts $\mathcal{A}_+$, $\mathcal{B}_+$, $\mathcal{C}_+$ and $\mathcal{N}_+$ mapping on the set Ω_+ and parts $\mathcal{A}_-$, $\mathcal{B}_-$, $\mathcal{C}_-$ and $\mathcal{N}_-$ acting on Ω_-^δ. By use of (7), the forms $\mathcal{A}_-$, $\mathcal{B}_-$, $\mathcal{C}_-$ and $\mathcal{N}_-$ lead to scaled forms $\mathcal{A}_-^\delta$, $\mathcal{B}_-^\delta$, $\mathcal{C}_-^\delta$ and $\mathcal{N}_-^\delta$ mapping on the fixed domain Ω_-, but depending on the parameter δ. Indeed, negative powers of δ appear in their formulations providing a scaled variational problem that can be seen as a "stiff problem".

1.2. The limit problem

Proposition 1.1. *Suppose that the scaled initial energy $E^\delta(0)$ is bounded independently of δ. Then*

- *w^δ, ϕ^δ, ψ^δ and $(w^\delta)'$, $(\psi^\delta)'$, $(\phi^\delta)'$ are bounded independently of δ in $L^\infty(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ respectively;*

- u^δ *and* $(u^\delta)'$ *are bounded independently of* δ *in* $L^\infty(0,T;(H^1(\Omega))^2)$ *and* $L^\infty(0,T;(L^2(\Omega))^2)$ *respectively;*
- $\delta^{-1}\partial_z\phi^\delta_-$, $\delta^{-1}\partial_z\psi^\delta_-$, $\delta^{-1}\partial_z w^\delta_-$, $(\partial_x u^\delta_{-1}+1\backslash 2(\partial_x w^\delta_-)^2)$, $\delta^{-1}(\partial_z u^\delta_{-1}+\partial_x u^\delta_{-2}+\partial_x w^\delta_-\partial_z w^\delta_-)$ *and* $\delta^{-2}(\partial_z u^\delta_{-2}+1\backslash 2(\partial_z w^\delta_-)^2)$ *are bounded independently of* δ *in* $L^\infty(0,T;(L^2(\Omega_-)))$.

As a consequence of the above proposition, we can assert that there exist subsequences (still noted by w^δ, ψ^δ, u^δ and ϕ^δ) such that $w^\delta\to\widetilde{w}$, $\psi^\delta\to\widetilde{\psi}$, $\phi^\delta\to\widetilde{\phi}$ and $u^\delta\to\widetilde{u}$ weakly $*$ in $L^\infty(0,T;H^1(\Omega))$ and $L^\infty(0,T;(H^1(\Omega))^2)$, respectively. Their time derivatives converge weakly $*$ respectively towards $\widetilde{w}'$, $\widetilde{\psi}'$, $\widetilde{\phi}'$ and $\widetilde{u}'$ in $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;(L^2(\Omega))^2)$ respectively. These limits are characterised by the following.

Proposition 1.2. *We have*:

- $\widetilde{w}_+,\widetilde{\phi}_+,\widetilde{\psi}_+\in L^\infty(0,T;V(\Omega_+))$; $\widetilde{w}'_+,\widetilde{\psi}'_+,\widetilde{\phi}'_+\in L^\infty(0,T;L^2(\Omega_+))$; $(\widetilde{w}_+|_\Sigma)',(\widetilde{\psi}_+|_\Sigma)',(\widetilde{\phi}_+|_\Sigma)'\in L^\infty(0,T;L^2(\Sigma))$,
- $\widetilde{u}_+\in L^\infty(0,T;U(\Omega_+))$, $\widetilde{u}'_+\in L^\infty(0,T;[L^2(\Omega_+)]^2)$, $(\widetilde{u}_+|_\Sigma)'\in L^\infty(0,T;(L^2(\Sigma))^2)$, *where*:

$$V(\Omega_+)=\{v\in H^1(\Omega_+);\ v|_{\Gamma_+}=0,\ v|_\Sigma\in H^1_0(\Sigma)\}\quad\text{and}$$
$$U(\Omega_+)=\{\varphi\in (H^1(\Omega_+))^2;\ \varphi|_{\Gamma_+}=0,\ \varphi_1|_\Sigma\in H^1_0(\Sigma)\}.$$

Moreover, $\widetilde{w}_-=\widetilde{w}_+|_\Sigma$, $\widetilde{\psi}_-=\widetilde{\psi}_+|_\Sigma$, $\widetilde{\phi}_-=\widetilde{\phi}_+|_\Sigma$, $\widetilde{u}_{-1}=\widetilde{u}_{+1}|_\Sigma$ *and* $\widetilde{u}_{-2}=0$.

Theorem 1.1. *Suppose that the scaled initial data satisfies the following weak convergences*:

- $\psi^{*\delta}_+$, $w^{*\delta}_+$, $\phi^{*\delta}_+$, $u^{*\delta}_{+1}$ *and* $u^{*\delta}_{+2}$ *converge weakly to* $\widetilde{\psi}^*_+$, $\widetilde{w}^*_+$, $\widetilde{\phi}^*_+$, $\widetilde{u}^*_{+1}$ *and* $\widetilde{u}^*_{+2}$ *in* $H^1(\Omega_+)$, *with* $\widetilde{u}^*_{+2}|_\Sigma=0$,
- $\int_{-1}^0\psi^{*\delta}_-\,\mathrm{d}z$, $\int_{-1}^0 w^{*\delta}_-\,\mathrm{d}z$, $\int_{-1}^0\phi^{*\delta}_-\,\mathrm{d}z$, $\int_{-1}^0 u^{*\delta}_{-1}\,\mathrm{d}z$ *and* $\int_{-1}^0 u^{*\delta}_{-2}\,\mathrm{d}z$ *converge to* $\widetilde{\psi}^*_+|_\Sigma$, $\widetilde{w}^*_+|_\Sigma$, $\widetilde{\phi}^*_+|_\Sigma$, $\widetilde{u}^*_{+1}|_\Sigma$ *and* 0 *in* $H^1(\Sigma)$,
- $\psi^{**\delta}_+$, $w^{**\delta}_+$, $\phi^{**\delta}_+$, $u^{**\delta}_{+1}$ *and* $u^{**\delta}_{+2}$ *converge to* $\widetilde{\psi}^{**}_+$, $\widetilde{w}^{**}_+$, $\widetilde{\phi}^{**}_+$, $\widetilde{u}^{**}_{+1}$ *and* $\widetilde{u}^{**}_{+2}$ *in* $L^2(\Omega_+)$,
- $\int_{-1}^0\psi^{**\delta}_-\,\mathrm{d}z$, $\int_{-1}^0 w^{**\delta}_-\,\mathrm{d}z$, $\int_{-1}^0\phi^{**\delta}_-\,\mathrm{d}z$ *converge to* $\widetilde{\psi}^{***}$, $\widetilde{w}^{***}$, $\widetilde{\phi}^{***}$ *in* $L^2(\Sigma)$ *and* $(\int_{-1}^0 u^{**\delta}_{-1}\,\mathrm{d}z,\ \int_{-1}^0\delta^{-1}u^{**\delta}_{-2}\,\mathrm{d}z)$ *converge to* $\tilde{u}^{***}$ *in* $L^2(\Sigma)\times L^2(\Sigma)$.

Then the subsequence u^δ_+ (*respectively* ψ^δ_+, w^δ_+, ϕ^δ_+) *converges weakly* $*$ *in* $L^\infty(0,T;U(\Omega_+))$ (*respectively in* $L^\infty(0,T;V(\Omega_+))$) *to* $\widetilde{u}_+$ (*respectively to* $\widetilde{\psi}_+$, $\widetilde{w}_+$, $\widetilde{\phi}_+$), *which satisfies the problem*:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_+\big(\widetilde{w}'_+,\widetilde{u}'_+,\widetilde{\psi}'_+,\widetilde{\phi}'_+,\hat{w},\hat{u},\hat{\psi},\hat{\phi}\big)+\mathcal{A}_+\big(\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{\psi},\hat{\phi}\big)+\mathcal{B}_+(\widetilde{w}_+,\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{w},\hat{\psi},\hat{\phi})+\mathcal{N}_+(\widetilde{u}_+,\widetilde{w}_+,\hat{u},\hat{w})\\ \quad+\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_\Sigma\big(\widetilde{w}'_+,\widetilde{u}'_+,\widetilde{\psi}'_+,\widetilde{\phi}'_+,\hat{w},\hat{u},\hat{\psi},\hat{\phi}\big)+\mathcal{A}_\Sigma(\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{\psi},\hat{\phi})+\mathcal{B}_\Sigma(\widetilde{w}_+,\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{w},\hat{\psi},\hat{\phi})\\ \quad+\mathcal{N}_\Sigma(\widetilde{u}_+,\widetilde{w}_+,\hat{u},\hat{w})=0,\end{cases}\tag{8}$$

$\forall(\hat{u},\hat{w},\hat{\psi},\hat{\phi})\in U(\Omega_+)\times[V(\Omega_+)]^3$ *with the initial conditions*:

$$\begin{aligned}&\widetilde{u}_+(0)=\widetilde{u}^*_+,\qquad \widetilde{\psi}_+(0)=\widetilde{\psi}^*_+,\qquad \widetilde{\phi}_+(0)=\widetilde{\phi}^*_+,\qquad \widetilde{w}_+(0)=\widetilde{w}^*_+\quad\text{in }\Omega_+,\\ &\widetilde{\psi}'_+(0)=\widetilde{\psi}^{**}_+,\qquad \widetilde{\phi}'_+(0)=\widetilde{\phi}^{**}_+,\qquad \widetilde{u}'_+(0)=\widetilde{u}^{**}_+,\qquad \widetilde{w}'_+(0)=\widetilde{w}^{**}_+\quad\text{in }\Omega_+,\\ &\widetilde{\psi}_+(0)=\widetilde{\psi}^*_+\big|_\Sigma,\qquad \widetilde{\phi}_+(0)=\widetilde{\phi}^*_+\big|_\Sigma,\qquad \widetilde{w}_+(0)=\widetilde{w}^*_+\big|_\Sigma,\qquad \widetilde{u}_{+1}(0)=\tilde{u}^*_{+1}\big|_\Sigma,\\ &\widetilde{u}_{+2}(0)=0,\qquad \widetilde{\psi}'_+(0)=\widetilde{\psi}^{***}\quad\text{on }\Sigma,\\ &\widetilde{\phi}'_+(0)=\widetilde{\phi}^{***},\qquad \widetilde{w}'_+(0)=\widetilde{w}^{***},\qquad \widetilde{u}'_+(0)=\widetilde{u}^{***}\quad\text{on }\Sigma,\end{aligned}\tag{9}$$

where $\mathcal{C}_\Sigma(\widetilde{w}'_+,\widetilde{u}'_+,\widetilde{\psi}'_+,\widetilde{\phi}'_+,\hat{w},\hat{u},\hat{\psi},\hat{\phi})=\rho_-\int_\Sigma[(\widetilde{w}_+|_\Sigma)'\hat{w}|_\Sigma+(\widetilde{u}_+|_\Sigma)'.\hat{u}|_\Sigma+(\widetilde{\psi}_+|_\Sigma)'\hat{\psi}|_\Sigma+(\hat{\phi}_+|_\Sigma)'\hat{\phi}|_\Sigma]\,\mathrm{d}x$, $\mathcal{A}_\Sigma(\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{\psi},\hat{\phi})=E_-\int_\Sigma(\partial_x\widetilde{\psi}_+|_\Sigma)(\partial_x\hat{\psi}|_\Sigma)\,\mathrm{d}x$, $\mathcal{B}_\Sigma(\widetilde{w}_+,\widetilde{\psi}_+,\widetilde{\phi}_+,\hat{w},\hat{\psi},\hat{\phi})=K_-\int_\Sigma(\widetilde{\psi}_+|_\Sigma+\partial_x\widetilde{w}_+|_\Sigma)(\hat{\psi}|_\Sigma+\partial_x\hat{w}|_\Sigma)\,\mathrm{d}x$, $\mathcal{N}_\Sigma(\widetilde{u}_+,\widetilde{w}_+,\hat{u},\hat{w})=E_-\int_\Sigma[\partial_x\widetilde{u}_{+1}|_\Sigma+\frac{1}{2}(\partial_x\widetilde{w}_+|_\Sigma)^2][\partial_x\hat{u}_{+1}|_\Sigma+(\partial_x\widetilde{w}_+|_\Sigma)(\partial_x\hat{w}|_\Sigma)]\,\mathrm{d}x$.

Proof. Using adequate test functions and taking advantage of Propositions 1.1 and 1.2, we go through the limit in the scaled variational problem as $\delta\to 0$. We show that the forms $\mathcal{A}^\delta_-$, $\mathcal{B}^\delta_-$, $\mathcal{C}^\delta_-$ and $\mathcal{N}^\delta_-$ mapping on the scaled domain Ω_- converge towards the forms $\mathcal{A}_\Sigma$, $\mathcal{B}_\Sigma$, $\mathcal{C}_\Sigma$ and $\mathcal{N}_\Sigma$, which map on the boundary Σ. The difficulties arising from the nonlinear terms are treated thanks to the Sobolev imbeddings: $H^1(\Omega)\underset{\text{compact}}{\hookrightarrow}H^{1-\varepsilon}(\Omega)\hookrightarrow L^{\frac{2}{\varepsilon}}(\Omega)$, $\forall\varepsilon>0$. □

The existence and the uniqueness of the solution of this problem may be shown by means of semi-group theory. This allows the convergence of the whole sequence $(u^\delta_+,\psi^\delta_+,w^\delta_+,\phi^\delta_+)$ to $(\widetilde{u}_+,\widetilde{\psi}_+,\widetilde{w}_+,\widetilde{\phi}_+)$.

Remark 1. Denoting by $\widetilde{N}_1$, $\widetilde{N}_{12}$ and $\widetilde{N}_2$ the expressions obtained from N_1, N_{12} and N_2 by replacing u, E and μ with $\widetilde{u}_+$, E_+ and μ_+; the limit problem is, formally, equivalent to the boundary value problem:

$$\begin{aligned}
&\rho_+\widetilde{u}''_{+1} - [\partial_x\widetilde{N}_1 + \partial_y\widetilde{N}_{12}] = 0, \qquad \rho_+\widetilde{u}''_{+2} - [\partial_y\widetilde{N}_2 + \partial_x\widetilde{N}_{12}] = 0,\\
&\rho_+\widetilde{w}''_+ - K_+\big[\partial_x(\partial_x\widetilde{w}_+ + \widetilde{\psi}_+) + \partial_y(\partial_y\widetilde{w}_+ + \widetilde{\phi}_+)\big] - \partial_x(\widetilde{N}_1\partial_x\widetilde{w}_+ + \widetilde{N}_{12}\partial_y\widetilde{w}_+) - \partial_y(\widetilde{N}_2\partial_y\widetilde{w}_+ + \widetilde{N}_{12}\partial_x\widetilde{w}_+) = 0,\\
&\rho_+\widetilde{\psi}''_+ - D_+\left[\partial_x^2\widetilde{\psi}_+ + \frac{1-\mu_+}{2}\partial_y^2\widetilde{\psi}_+ + \frac{1+\mu_+}{2}\partial_{xy}^2\widetilde{\phi}_+\right] + K_+[\widetilde{\psi}_+ + \partial_x\widetilde{w}_+] = 0,\\
&\rho_+\widetilde{\phi}''_+ - D_+\left[\partial_y^2\widetilde{\phi}_+ + \frac{1-\mu_+}{2}\partial_x^2\widetilde{\phi}_+ + \frac{1+\mu_+}{2}\partial_{xy}^2\widetilde{\psi}_+\right] + K_+[\widetilde{\phi}_+ + \partial_y\widetilde{w}_+] = 0 \quad \text{in } \Omega_+ \times (0,T),
\end{aligned} \tag{10}$$

with clamped boundary conditions:

$$\widetilde{u}_+ = 0, \qquad \widetilde{w}_+ = 0, \qquad \widetilde{\psi}_+ = 0, \qquad \widetilde{\phi}_+ = 0 \quad \text{on } \Gamma_+ \times (0,T), \tag{11}$$

and **Approximate boundary conditions on** Σ:

$$\begin{aligned}
&\nu_1\widetilde{N}_1 + \nu_2\widetilde{N}_{12} = -\rho_-(\widetilde{u}_{+1})'' + E_-\partial_x\left[\partial_x\tilde{u}_{+1} + \frac{1}{2}(\partial_x\widetilde{w}_+)^2\right], \qquad \nu_2\widetilde{N}_2 + \nu_1\widetilde{N}_{12} = -\rho_-(\widetilde{u}_{+2})'',\\
&D_+\left[\nu_1\partial_x\widetilde{\psi}_+ + \mu_+\nu_1\partial_y\widetilde{\phi}_+ + \frac{1-\mu_+}{2}[\partial_y\widetilde{\psi}_+ + \partial_x\widetilde{\phi}_+]\nu_2\right] = -\rho_-\widetilde{\psi}''_+ + E_-\partial_x^2\widetilde{\psi}_+ - K_-(\partial_x\widetilde{w}_+ + \widetilde{\psi}_+),\\
&D_+\left[\nu_2\partial_y\widetilde{\phi}_+ + \mu_+\nu_2\partial_x\widetilde{\psi}_+ + \frac{1-\mu_+}{2}[\partial_y\widetilde{\psi}_+ + \partial_x\widetilde{\phi}_+]\nu_1\right] = -\rho_-\widetilde{\phi}''_+,\\
&K_+[\partial_\nu\widetilde{w}_+ + \nu_1\widetilde{\psi}_+ + \nu_2\widetilde{\phi}_+] + (\nu_1\widetilde{N}_1 + \nu_2\widetilde{N}_{12})\partial_x\widetilde{w}_+ + (\nu_2\widetilde{N}_2 + \nu_1\widetilde{N}_{12})\partial_y\widetilde{w}_+\\
&\quad = -\rho_-\widetilde{w}''_+ + K_-\partial_x(\partial_x\widetilde{w}_+ + \widetilde{\psi}_+) + E_-\partial_x\left[\left(\partial_x\widetilde{u}_{+1} + \frac{1}{2}(\partial_x\widetilde{w}_+)^2\right)\partial_x\widetilde{w}_+\right],
\end{aligned} \tag{12}$$

with the initial conditions (9).

Note that the assumption made on the characteristics of the stiffener (they behave as δ^{-1}) needs to be assumed in order to obtain the model described above, which can be viewed as a zero-order approximation of the original problem. If both the plate's and Stiffener's characteristics are independent of δ or of the same asymptotic order, we obtain, as $\delta \to 0$, a problem where the effect of the thin stiffener is completely neglected. A higher-order approximation is thus needed to approach the effect of the stiffener.

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