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Functional analysis

Vertical symbols, Toeplitz operators on weighted Bergman spaces over the upper half-plane and very slowly oscillating functions

Les symboles verticaux, opérateurs de Toeplitz sur les espaces pondérés de Bergman sur le demi-plan supérieur et fonctions à oscillation très lente

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ABSTRACT

Extending a recent result of Herrera Yañez, Maximenko and Vasilevski, we provide a further step in the structural analysis of algebras generated by Toeplitz operators on weighted Bergman spaces over the upper half-plane. We show that the set of "spectral" functions corresponding to Toeplitz operators generated by bounded vertical symbols is dense in the C^*-algebra of very slowly oscillating functions on the positive half-line.

RÉSUMÉ

En étendant le résultat récent de Herrera Yañez, Maximenko et Vasilevski, nous allons proposer une nouvelle étape dans l'analyse structurelle des algèbres générées par les opérateurs de Toeplitz agissant sur les espaces pondérés de Bergman sur le demi-plan supérieur. Nous allons montrer que l'ensemble des fonctions « spectrales » correspondant aux opérateurs de Toeplitz à symboles bornés verticaux est dense dans la C*-algèbre des fonctions à oscillation très lente sur la demi-droite positive.

1. Introduction and description of results

The paper is devoted to the description of a class of Toeplitz operators acting on the weighted Bergman spaces (depending on a real parameter $\lambda \in (-1, +\infty)$) over the upper half-plane and of the C*-algebra generated by them. For the upper half-plane $\Pi = \{w = u + \mathrm{i}v \in \mathbb{C}; v > 0\}$ and the canonical weights $\rho_\lambda(w) = \frac{\lambda+1}{\pi}(2\,\mathrm{Im}(w))^\lambda$, let $\mathrm{d}\mu_\lambda = \rho_\lambda\,\mathrm{d}\mu$ be the weighted measure on Π with $\mathrm{d}\mu$ being the Lebesgue plane measure on Π. The weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ of analytic functions is a (closed) subspace of $L_2(\Pi, \mathrm{d}\mu_\lambda)$ consisting of measurable functions f on Π for which the norm $\|f\|_{L_2(\Pi,\mathrm{d}\mu_\lambda)} = (\int_\Pi |f(w)|^2\,\mathrm{d}\mu_\lambda(w))^{1/2}$ is finite. Given a function (generating symbol) $g \in L_\infty(\Pi)$, the Toeplitz operator (TO,

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for short) $T_g^{(\lambda)}$ acting on $\mathcal{A}_\lambda^2(\Pi)$ is defined as $T_g^{(\lambda)} f = B_{\Pi,\lambda}(gf)$, where $B_{\Pi,\lambda}$ is the orthogonal projection of $L_2(\Pi, \mathrm{d}\mu_\lambda)$ onto $\mathcal{A}_\lambda^2(\Pi)$. A time-frequency approach to TO's acting on (poly)analytic function spaces is described in [3].

A measurable function $g : \Pi \to \mathbb{C}$ will be called *vertical* if there exists a measurable function $a : \mathbb{R}_+ \to \mathbb{C}$ such that $g(w) = a(\mathrm{Im}(w))$ for almost all $w \in \Pi$. The key result giving an easy access to the properties of TO's acting on $\mathcal{A}_\lambda^2(\Pi)$ with bounded vertical symbols was established by Vasilevski and his collaborators: see [5] for weightless case, [1] for weighted case, and the summarizing book [6]. Namely, it is proved that the Toeplitz operator $T_{a\circ\mathrm{Im}}^{(\lambda)}$ acting on $\mathcal{A}_\lambda^2(\Pi)$ with $a \in L_\infty(\mathbb{R}_+)$ is unitarily equivalent to the multiplication operator $M_{\gamma_{a,\lambda}}$ acting on $L_2(\mathbb{R}_+)$, where:

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^{+\infty} a\left(\frac{v}{2}\right) v^\lambda \mathrm{e}^{-xv}\, \mathrm{d}v.$$

The "spectral" function $\gamma_{a,\lambda}$ carries many substantial properties of the corresponding Toeplitz operator $T_{a\circ\mathrm{Im}}^{(\lambda)}$. In particular, each C*-algebra $\mathcal{T}_\lambda$ generated by TO's with bounded vertical symbols is commutative and is isometrically isomorphic to the C*-algebra generated by the set:

$$\mathfrak{S}_\lambda = \mathfrak{S}_\lambda\big(L_\infty(\mathbb{R}_+)\big) = \big\{\gamma_{a,\lambda};\ a \in L_\infty(\mathbb{R}_+)\big\} \subset L_\infty(\mathbb{R}_+).$$

We analyze here the corresponding properties of the following two C*-algebras: the C*-algebra $\mathcal{T}_\lambda$ generated by TO's with bounded vertical symbols and the C*-algebra generated by the set $\mathfrak{S}_\lambda$ of the "spectral" functions. The first step in this direction was done by Suárez [4] dealing with TO's on the unit disk with bounded radial symbols. Suárez's results influenced the recent work [2] where it is proved that the set $\mathfrak{S}_0$ *is dense in the* C*-*algebra* $\mathrm{VSO}(\mathbb{R}_+)$ *of very slowly oscillating functions on* $\mathbb{R}_+$, i.e., the functions uniformly continuous with respect to the logarithmic distance $\rho(x, y) = |\log x - \log y|$ on $\mathbb{R}_+$.

Following the approach from [2], we show that *the* C*-*algebra generated by* $\mathfrak{S}_\lambda$ *does not actually depend on* λ, *and coincides with* $\mathrm{VSO}(\mathbb{R}_+)$. For that reason, introduce (a parametric family of) distances on $\mathbb{R}_+$:

$$\varkappa_\lambda(x, y) = \sup\big\{\big|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)\big|;\ a \in L_\infty(\mathbb{R}_+),\ \|a\|_\infty \leqslant 1\big\}.$$

In other words, $\varkappa_\lambda$ is the smallest distance Λ such that $|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)| \leqslant \|a\|_\infty \Lambda(x, y)$ uniformly in a. As our first result, we prove the following:

Theorem 1. *Each distance* $\varkappa_\lambda$ *is uniformly equivalent to the logarithmic distance* ρ.

In particular, the inequality $|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)| \leqslant 2(\lambda+1)\|a\|_\infty \rho(x, y)$ holds for each $a \in L_\infty(\mathbb{R}_+)$ and for all $x, y > 0$, which means that each function $\gamma_{a,\lambda}$ is Lipschitz continuous with respect to ρ, and thus $\gamma_{a,\lambda} \in \mathrm{VSO}(\mathbb{R}_+)$. Then the main result of this paper generalizing that of [2, Theorem 5.5] reads as follows.

Theorem 2. *Each* $\mathfrak{S}_\lambda$ *is a dense subset of* $\mathrm{VSO}(\mathbb{R}_+)$.

It follows that there is no qualitative difference between the weightless and weighted cases when studying the structure of commutative algebra $\mathcal{T}_\lambda$. Indeed, each C*-algebra $\mathcal{T}_\lambda$ generated by TO's with bounded vertical symbols coincides with the closure of the set of these TO's. Furthermore, the algebras $\mathcal{T}_\lambda$ for any $\lambda \in (-1, +\infty)$ are all isomorphic and isometric among each other, being isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

2. Sketch of proof of Theorem 1: Lipschitz continuity of "spectral" functions

For each $x, y, v > 0$, denote by $D_\lambda(x, y, v)$ the difference of the "kernels" corresponding to x and y:

$$D_\lambda(x, y, v) = \frac{x^{\lambda+1}\mathrm{e}^{-xv} v^\lambda}{\Gamma(\lambda+1)} - \frac{y^{\lambda+1}\mathrm{e}^{-yv} v^\lambda}{\Gamma(\lambda+1)}.$$

Then for every function $a \in L_\infty(\mathbb{R}_+)$ we have the following simple estimate:

$$\big|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)\big| \leqslant \|a\|_\infty \int_0^\infty \big|D_\lambda(x, y, v)\big|\, \mathrm{d}v.$$

If x and y are fixed, $x \neq y$, and a is defined in such a manner that $a(v/2)$ is equal to the sign of $D_\lambda(x, y, v)$, then $\|a\|_\infty = 1$ and $|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)| = \int_0^\infty |D_\lambda(x, y, v)|\, \mathrm{d}v$. Therefore,

$$\varkappa_\lambda(x, y) = \int_0^\infty \big|D_\lambda(x, y, v)\big|\, \mathrm{d}v. \tag{1}$$

The expression $D_\lambda(x,y,v)$ changes its sign at the point $v_0=(\lambda+1)(\log(x)-\log(y))/(x-y)$. Dividing the integral (1) into two integrals over $(0,v_0]$ and $(v_0,+\infty)$, we can write it in terms of the upper incomplete Gamma function:

$$\varkappa_\lambda(x,y)=\frac{2}{\Gamma(\lambda+1)}\left|\Gamma\left(\lambda+1,\frac{(\lambda+1)x\log(y/x)}{y-x}\right)-\Gamma\left(\lambda+1,\frac{(\lambda+1)y\log(y/x)}{y-x}\right)\right|.$$

Note that $\varkappa_\lambda$ may be expressed through ρ via formula:

$$\varkappa_\lambda(x,y)=\xi_\lambda\big(\rho(x,y)\big), \tag{2}$$

where ξ_λ is defined on $[0,+\infty)$ by:

$$\xi_\lambda(\delta)=\frac{2}{\Gamma(\lambda+1)}\left(\Gamma\left(\lambda+1,\frac{(\lambda+1)\delta}{e^\delta-1}\right)-\Gamma\left(\lambda+1,\frac{(\lambda+1)\delta}{1-e^{-\delta}}\right)\right).$$

Each function ξ_λ maps $[0,+\infty)$ onto $[0,2)$. Moreover, it is strictly increasing and strictly concave on $\mathbb{R}_+$. The derivative of ξ_λ is bounded from above by $\xi'_\lambda(0)$, the latter being:

$$\xi'_\lambda(0)=\frac{2(1+\lambda)^{1+\lambda}}{\Gamma(1+\lambda)\,e^{1+\lambda}}.$$

Formula (2) and the properties of ξ_λ imply that $\varkappa_\lambda$ is finite and uniformly equivalent to ρ. More precisely, $\varkappa_\lambda(x,y)/\rho(x,y)\to\xi'_\lambda(0)$ as $\rho(x,y)\to 0$ and $\varkappa_\lambda(x,y)\leqslant\xi'_\lambda(0)\rho(x,y)$ for all $x,y\in\mathbb{R}_+$. The latter inequality is more exact than $|\gamma_{a,\lambda}(x)-\gamma_{a,\lambda}(y)|\leqslant 2(\lambda+1)\|a\|_\infty\rho(x,y)$, which can be deduced directly from (1).

3. Sketch of proof of Theorem 2: density of $\mathfrak{S}_\lambda$ in VSO($\mathbb{R}_+$)

For each $\lambda\in(-1,+\infty)$, introduce the sequences of functions $\psi_{n,\lambda}:\mathbb{R}_+\to\mathbb{C}$ and $\omega_{n,\lambda}:\mathbb{R}_+\to\mathbb{C}$ by

$$\psi_{n,\lambda}(x)=\frac{1}{\mathrm{B}(n+\lambda,n+\lambda)}\frac{x^{n+\lambda}}{(1+x)^{2(n+\lambda)}}\quad\text{and}\quad\omega_{n,\lambda}(x)=\frac{1}{[\Gamma(n+\lambda)]^2}\frac{d^{n-1}}{dx^{n-1}}\big(e^{-x}x^{2(n+\lambda)-1}\big),\quad n\in\mathbb{N},$$

respectively, where B is the Euler Beta function. As in [2, Proposition 5.1] it may be proved that for each $\lambda\in(-1,+\infty)$ the sequence $(\psi_{n,\lambda})_{n=1}^\infty$ is a Dirac sequence on the multiplicative group $\mathbb{R}_+$. Also, it can easily be verified that the functions $\psi_{n,\lambda}$ and $\omega_{n,\lambda}$ are connected via the Laplace transform $(\mathcal{L}f)(x)=\int_0^\infty f(t)\,e^{-xt}\,dt$ as follows: $\psi_{n,\lambda}(x)=x^{\lambda+1}(\mathcal{L}\omega_{n,\lambda})(x)$ for each $\lambda\in(-1,+\infty)$ and each $n\in\mathbb{N}$.

Further, given a function $f:\mathbb{R}_+\to\mathbb{C}$ put $\tilde f(x)=f(1/x)$. Then the mapping $f\mapsto\tilde f$ is an involution, and for all $f\in L_\infty(\mathbb{R}_+)$ and $g\in L_1(\mathbb{R}_+,\frac{dx}{x})$, the identity $\widetilde{f\star g}=\tilde f\star\tilde g$ holds, where:

$$(f\star g)(x)=\int_0^\infty f(y)\,g\left(\frac{x}{y}\right)\frac{dy}{y},\quad x\in\mathbb{R}_+,$$

is the multiplicative (Mellin) convolution on $\mathbb{R}_+$. For a generating symbol $a\in L_\infty(\mathbb{R}_+)$ each function $\gamma_{a,\lambda}$ may be written in the form of a Mellin convolution as follows:

$$\gamma_{a,\lambda}(x)=\frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty a\left(\frac{v}{2}\right)v^\lambda e^{-xv}\,dv=\frac{1}{\Gamma(\lambda+1)}\int_0^\infty a(v)(2xv)^{\lambda+1}e^{-2xv}\frac{dv}{v}=(\tilde a\star K_\lambda)(x),$$

where the function K_λ is defined on $\mathbb{R}_+$ by:

$$K_\lambda(z)=\frac{(2z)^{\lambda+1}e^{-2z}}{\Gamma(\lambda+1)}.$$

Then for the function $\phi_{n,\lambda}(x)=\Gamma(\lambda+1)(2x)^{-\lambda}\omega_{n,\lambda}(2x)$, we have:

$$\gamma_{\phi_{n,\lambda},\lambda}(x)=(\widetilde{\phi_{n,\lambda}}\star K_\lambda)(x)=2x^{\lambda+1}\int_0^\infty\omega_{n,\lambda}(2v)\,e^{-2xv}\,dv=x^{\lambda+1}(\mathcal{L}\,\omega_{n,\lambda})(x)=\psi_{n,\lambda}(x),$$

i.e., each function $\psi_{n,\lambda}$ is a certain "spectral" function. To finish the proof of Theorem 2, define the symbol $a_{n,\lambda}:\mathbb{R}_+\to\mathbb{C}$ by $a_{n,\lambda}=\tilde\sigma\star\phi_{n,\lambda}$ with $\sigma\in\mathrm{VSO}(\mathbb{R}_+)$. Since $\phi_{n,\lambda}\in L_1(\mathbb{R}_+,\frac{dx}{x})$, then $a_{n,\lambda}\in L_\infty(\mathbb{R}_+)$, and therefore

$$\mathfrak{S}_\lambda\ni\gamma_{a_{n,\lambda}\lambda}=\widetilde{a_{n,\lambda}}\star K_\lambda=(\tilde{\tilde\sigma}\star\widetilde{\phi_{n,\lambda}})\star K_\lambda=\sigma\star(\widetilde{\phi_{n,\lambda}}\star K_\lambda)=\sigma\star\psi_{n,\lambda}.$$

Then, from the well-known general fact on Dirac sequences and uniformly continuous functions on locally compact groups, we get $\lim_{n\to\infty}\|\sigma\star\psi_{n,\lambda}-\sigma\|_\infty=0$ for each $\lambda\in(-1,+\infty)$, which completes the proof.

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References

[1] S. Grudsky, A. Karapetyants, N. Vasilevski, Dynamics of properties of Toeplitz operators on the upper half-plane: Parabolic case, J. Oper. Theory 52 (2004) 185–204.
[2] C. Herrera Yañez, E.A. Maximenko, N.L. Vasilevski, Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions, Integral Equ. Oper. Theory 77 (2013) 149–166.
[3] O. Hutník, M. Hutníková, Toeplitz operators on poly-analytic spaces via time-scale analysis, Oper. Matrices (2013), in press.
[4] D. Suárez, The eigenvalues of limits of radial Toeplitz operators, Bull. Lond. Math. Soc. 40 (2008) 631–641.
[5] N.L. Vasilevski, On Bergman–Toeplitz operators with commutative symbol algebras, Integral Equ. Oper. Theory 34 (1999) 107–126.
[6] N.L. Vasilevski, Commutative Algebras of Toeplitz Operators on the Bergman Space, Operator Theory: Advances and Applications, vol. 185, Birkhäuser, Basel, 2008.