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Restrictions of Brownian motion

Restrictions du mouvement brownien

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ABSTRACT

Let $\{B(t): 0 \leq t \leq 1\}$ be a linear Brownian motion and let dim denote the Hausdorff dimension. Let $\alpha > \frac{1}{2}$ and $1 \leq \beta \leq 2$. We prove that, almost surely, there exists no set $A \subset [0,1]$ such that $\dim A > \frac{1}{2}$ and $B: A \to \mathbb{R}$ is α-Hölder continuous. The proof is an application of Kaufman's dimension doubling theorem. As a corollary of the above theorem, we show that, almost surely, there exists no set $A \subset [0,1]$ such that $\dim A > \frac{\beta}{2}$ and $B: A \to \mathbb{R}$ has finite β-variation. The zero set of B and a deterministic construction witness that the above theorems give the optimal dimensions.

RÉSUMÉ

On note $\{B(t): 0 \leq t \leq 1\}$ un mouvement brownien linéaire et dim la dimension de Hausdorff. Pour $\alpha > \frac{1}{2}$ et $1 \leq \beta \leq 2$, nous montrons que, presque sûrement, il n'existe pas d'ensemble $A \subset [0,1]$ tel que $\dim A > \frac{1}{2}$ et $B: A \to \mathbb{R}$ soit α-Hölder continue. La preuve est une application du théorème de Kaufman sur le doublement de dimension. Comme corollaire du théorème ci-dessus, nous montrons que, presque sûrement, il n'existe pas d'ensemble $A \subset [0,1]$ tel que $\dim A > \frac{\beta}{2}$ et $B: A \to \mathbb{R}$ ait une β-variation finie. L'ensemble des zéros de B et une construction déterministe montrent que les théorèmes ci-dessus donnent les dimensions optimales.

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1. Introduction

We examine how large a set can be, on which linear Brownian motion is α-Hölder continuous for some $\alpha > \frac{1}{2}$ or has finite β-variation for some $1 \leq \beta \leq 2$. The main goal of the paper is to prove the following two theorems.

Theorem 1.1. *Let $\{B(t): 0 \leq t \leq 1\}$ be a linear Brownian motion and let $\alpha > \frac{1}{2}$. Then, almost surely, there exists no set $A \subset [0,1]$ with $\dim A > \frac{1}{2}$ such that $B: A \to \mathbb{R}$ is α-Hölder continuous.*

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Recall that for $A \subset [0,1]$ the *β-variation* of a function $f : A \to \mathbb{R}$ is defined as

$$\operatorname{Var}^\beta(f) = \sup\left\{\sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right|^\beta : x_0 < \cdots < x_n,\ x_i \in A,\ n \in \mathbb{N}^+\right\}.$$

Theorem 1.2. *Let $\{B(t): 0 \le t \le 1\}$ be a linear Brownian motion and assume that $1 \le \beta \le 2$. Then, almost surely, there exists no set $A \subset [0,1]$ with $\dim A > \frac{\beta}{2}$ such that $B: A \to \mathbb{R}$ has finite β-variation. In particular,*

$$\mathbb{P}\left(\exists A : \dim A > \frac{1}{2} \text{ and } B|_A \text{ is increasing}\right) = 0.$$

Clearly, the above theorems hold simultaneously for a countable dense set of parameters α, β, thus simultaneously for all α, β. Let $\mathcal{Z}$ be the zero set of a linear Brownian motion B. Then, almost surely, $\dim \mathcal{Z} = \frac{1}{2}$ and $B|_{\mathcal{Z}}$ is α-Hölder continuous for all $\alpha > \frac{1}{2}$, so Theorem 1.1 gives the optimal dimension. We prove also that Theorem 1.2 is best possible, see Theorem 4.3.

1.1. Motivation and related results

Let $C[0,1]$ denote the set of continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the maximum norm. Elekes proved the following restriction theorem.

Theorem 1.3. *(See Elekes [3].) Let $0 < \alpha < 1$. For the generic continuous function $f \in C[0,1]$ (in the sense of Baire category)*

(1) *for all $A \subset [0,1]$, if $f|_A$ is α-Hölder continuous, then $\dim A \le 1 - \alpha$;*
(2) *for all $A \subset [0,1]$, if $f|_A$ is of bounded variation, then $\dim A \le \frac{1}{2}$.*

The above theorem is sharp, the following result was proved by Kahane and Katznelson, and Máthé independently, by different methods.

Theorem 1.4. *(See Kahane and Katznelson [6], Máthé [10].) Let $0 < \alpha < 1$. For any $f \in C[0,1]$ there are compact sets $A, D \subset [0,1]$ such that*

(1) *$\dim A = 1 - \alpha$ and $f|_A$ is α-Hölder continuous;*
(2) *$\dim D = \frac{1}{2}$ and $f|_D$ is of bounded variation.*

Kahane and Katznelson also considered Hölder continuous functions.

Definition 1.5. For $A \subset [0,1]$ let $C^\alpha(A)$ and $BV(A)$ denote the set of functions $f : A \to \mathbb{R}$ that are α-Hölder continuous and of bounded variation, respectively. For all $0 < \alpha < \beta < 1$, define

$$H(\alpha,\beta) = \sup\left\{\gamma : \forall f \in C^\alpha[0,1]\ \exists A \subset [0,1] \text{ s.t. } \dim A = \gamma \text{ and } f|_A \in C^\beta(A)\right\},$$
$$V(\alpha) = \sup\left\{\gamma : \forall f \in C^\alpha[0,1]\ \exists A \subset [0,1] \text{ s.t. } \dim A = \gamma \text{ and } f|_A \in BV(A)\right\}.$$

Theorem 1.6. *(See Kahane and Katznelson [6].) For all $0 < \alpha < \beta < 1$, we have:*

$$H(\alpha,\beta) \le \frac{1-\beta}{1-\alpha} \quad \text{and} \quad V(\alpha) \le \frac{1}{2-\alpha}.$$

Question 1.7. *(See Kahane and Katznelson [6].) Is the above result the best possible?*

As the linear Brownian motion B is α-Hölder continuous for all $\alpha < \frac{1}{2}$, our results and Theorem 1.4 imply the following corollary.

Corollary 1.8. *For all $0 < \alpha < \frac{1}{2} < \beta < 1$ we have:*

$$H(\alpha,\beta) \le \frac{1}{2} \quad \text{and} \quad V(\alpha) = \frac{1}{2}.$$

Related results in the discrete setting can be found in [1].

Definition 1.9. Let $d \geq 2$ and $f:[0,1] \to \mathbb{R}^d$. We say that f is *increasing* on a set $A \subset [0,1]$ if all the coordinate functions of $f|_A$ are non-decreasing.

Question 1.10. *Let $d \geq 2$ and let $\{B(t): 0 \leq t \leq 1\}$ be a standard d-dimensional Brownian motion. What is the maximal number γ such that, almost surely, B is increasing on some set of Hausdorff dimension γ?*

2. Preliminaries

The diameter of a metric space X is denoted by $\operatorname{diam} X$. For all $s \geq 0$, the *s-dimensional Hausdorff measure* of X is defined as:

$$\mathcal{H}^s(X) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(X), \quad \text{where}$$
$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} X_i)^s : X \subset \bigcup_{i=1}^{\infty} X_i,\ \forall i\ \operatorname{diam} X_i \leq \delta\right\}.$$

The *Hausdorff dimension* of X is defined as:

$$\dim X = \inf\{s \geq 0 : \mathcal{H}^s(X) < \infty\}.$$

Let $A \subset \mathbb{R}$ and $\alpha > 0$. A function $f: A \to \mathbb{R}$ is called *α-Hölder continuous* if there exists a constant $c \in (0,\infty)$ such that $|f(x) - f(y)| \leq c|x-y|^\alpha$ for all $x, y \in A$.

Fact 2.1. *If $f: A \to \mathbb{R}$ is α-Hölder continuous, then* $\dim f(A) \leq \frac{1}{\alpha} \dim A$.

3. Hölder restrictions

The goal of this section is to prove Theorem 1.1. First we need some preparation.

Definition 3.1. A function $g:[0,1] \to \mathbb{R}^2$ is called *dimension doubling* if

$$\dim g(A) = 2 \dim A \quad \text{for all } A \subset [0,1].$$

Theorem 3.2. *(See Kaufman [7], see also [12].) The two-dimensional Brownian motion is almost surely dimension doubling.*

The following theorem follows from [5, Lemma 2] together with the fact that the closed range of the stable subordinator with parameter $\frac{1}{2}$ coincides with the zero set of a linear Brownian motion. For a more direct reference see [8].

Theorem 3.3. *Let $A \subset [0,1]$ be a compact set with* $\dim A > \frac{1}{2}$ *and let $\mathcal{Z}$ be the zero set of a linear Brownian motion. Then* $\dim(A \cap \mathcal{Z}) > 0$ *with positive probability.*

Lemma 3.4 *(Key Lemma). Let $\{W(t): 0 \leq t \leq 1\}$ be a linear Brownian motion. Assume that $\alpha > \frac{1}{2}$ and $f:[0,1] \to \mathbb{R}$ is a continuous function such that (f, W) is almost surely dimension doubling. Then there is no set $A \subset [0,1]$ such that* $\dim A > \frac{1}{2}$ *and f is α-Hölder continuous on A.*

Proof. Assume to the contrary that there is a set $A \subset [0,1]$ such that $\dim A > \frac{1}{2}$ and f is α-Hölder continuous on A. As f is still α-Hölder continuous on the closure of A, we may assume that A itself is closed. Let $\mathcal{Z}$ be the zero set of W, then Theorem 3.3 implies that $\dim(A \cap \mathcal{Z}) > 0$ with positive probability. Then the α-Hölder continuity of $f|_A$ and Fact 2.1 imply that, with positive probability,

$$\begin{aligned}\dim(f, W)(A \cap \mathcal{Z}) &= \dim\big(f(A \cap \mathcal{Z}) \times \{0\}\big) = \dim f(A \cap \mathcal{Z}) \\ &\leq \frac{1}{\alpha} \dim(A \cap \mathcal{Z}) < 2 \dim(A \cap \mathcal{Z}),\end{aligned}$$

which contradicts the fact that (f, W) is almost surely dimension doubling. □

Proof of Theorem 1.1. Let $\{W(t): 0 \leq t \leq 1\}$ be a linear Brownian motion which is independent of B. By Kaufman's dimension doubling theorem (B, W) is dimension doubling with probability one, thus applying Lemma 3.4 for an almost sure path of B finishes the proof. □

4. Restrictions of bounded variation

We need the following lemma, which may be obtained by a slight modification of [2, Lemma 4.1]. For the reader's convenience, we outline the proof.

Lemma 4.1. *Let $\alpha, \beta > 0$. Assume that $A \subset [0,1]$ and the function $f: A \to \mathbb{R}$ has finite β-variation. Then there are sets $A_n \subset A$ such that for any $n \in \mathbb{N}^+$*

$$f|_{A_n} \text{ is } \alpha\text{-Hölder continuous and } \dim\left(A \setminus \bigcup_{n=1}^{\infty} A_n\right) \le \alpha\beta.$$

Proof. For all $n \in \mathbb{N}^+$ let

$$A_n = \left\{x \in A : \left|f(x+t) - f(x)\right| \le 2t^{\alpha} \quad \text{for all } t \in [0, 1/n] \cap (A - x)\right\}.$$

As A is bounded, $f|_{A_n}$ is α-Hölder continuous for all $n \in \mathbb{N}^+$. Let

$$D = \left\{x \in A : \limsup_{t \to 0+} \left|f(x+t) - f(x)\right| t^{-\alpha} > 1\right\}.$$

Clearly $A \setminus \bigcup_{n=1}^{\infty} A_n \subset D$, so it is enough to prove that $\dim D \le \alpha\beta$. Let us fix $\delta > 0$ arbitrarily. Then for all $x \in D$ there is a $0 < t_x < \delta$ such that

$$\left|f(x+t_x) - f(x)\right| \ge t_x^{\alpha}. \tag{4.1}$$

Define $I_x = [x - t_x, x + t_x]$ for all $x \in D$. By Besicovitch's covering theorem (see [11, Thm. 2.7]) there is a number $p \in \mathbb{N}^+$ not depending on δ and countable sets $S_i \subset D$ ($i \in \{1, \dots, p\}$) such that

$$D \subset \bigcup_{i=1}^{p} \bigcup_{x \in S_i} I_x \quad \text{and} \quad I_x \cap I_y = \emptyset \quad \text{for all } x, y \in S_i,\ x \neq y. \tag{4.2}$$

Applying (4.1) and (4.2) implies that for all $i \in \{1, \dots, p\}$

$$\sum_{x \in S_i} |I_x|^{\alpha\beta} = 2^{\alpha\beta} \sum_{x \in S_i} t_x^{\alpha\beta} \le 2^{\alpha\beta} \sum_{x \in S_i} \left|f(x+t_x) - f(x)\right|^{\beta} \le 2^{\alpha\beta}\, \mathrm{Var}^{\beta}(f). \tag{4.3}$$

Eqs. (4.2) and (4.3) imply that

$$\mathcal{H}_{\delta}^{\alpha\beta}(D) \le \sum_{i=1}^{p} \sum_{x \in S_i} |I_x|^{\alpha\beta} \le p 2^{\alpha\beta}\, \mathrm{Var}^{\beta}(f).$$

As $\mathrm{Var}^{\beta}(f) < \infty$ and $\delta > 0$ was arbitrary, we obtain that $\mathcal{H}^{\alpha\beta}(D) < \infty$. Hence $\dim D \le \alpha\beta$, and the proof is complete. □

Proof of Theorem 1.2. Assume to the contrary that for some $\varepsilon > 0$ there is a random set $A \subset [0,1]$ such that, with positive probability, $\dim A \ge \beta/2 + 2\varepsilon$ and $B|_A$ has finite β-variation. Let $\alpha = 1/2 + \varepsilon/\beta > 1/2$. Applying Lemma 4.1 we obtain that there are sets $A_n \subset A$ such that $B|_{A_n}$ is α-Hölder continuous for every $n \in \mathbb{N}^+$ and

$$\dim\left(A \setminus \bigcup_{n=1}^{\infty} A_n\right) \le \alpha\beta = \frac{\beta}{2} + \varepsilon. \tag{4.4}$$

As $\alpha > 1/2$ and $B|_{A_n}$ are α-Hölder continuous, Theorem 1.1 implies that almost surely $\dim A_n \le 1/2$ for all $n \in \mathbb{N}^+$, therefore (4.4) and the countable stability of the Hausdorff dimension yield that $\dim A \le \beta/2 + \varepsilon$ almost surely. This is a contradiction, and the proof is complete. □

Theorems 4.2 and 4.3 (with $\alpha = \frac{1}{2}$) imply that Theorem 1.2 is sharp for all β.

Theorem 4.2. *(See Lévy's modulus of continuity, [9], see also [12].) For the linear Brownian motion $\{B(t): 0 \le t \le 1\}$, almost surely,*

$$\limsup_{h \to 0+} \sup_{0 \le t \le 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} = 1.$$

Theorem 4.3. *Let $0 < \alpha \le 1$ and $0 < \beta \le \frac{1}{\alpha}$ be fixed. Then there is a compact set $A \subset [0,1]$ such that $\dim A = \alpha\beta$ and if $f:[0,1] \to \mathbb{R}$ is a function and $c \in (0,\infty)$ such that for all $x, y \in [0,1]$*

$$\left|f(x) - f(y)\right| \le c|x-y|^\alpha \log \frac{1}{|x-y|}, \tag{4.5}$$

then $f|_A$ has finite β-variation.

Proof. First we construct A. For all $n \in \mathbb{N}$ let

$$\gamma_n = 2^{-n/(\alpha\beta)}(n+1)^{-(\beta+2)/\beta}.$$

We define intervals $I_{i_1\dots i_n} \subset [0,1]$ for all $n \in \mathbb{N}$ and $\{i_1,\dots,i_n\} \in \{0,1\}^n$ by induction. We use the convention $\{0,1\}^0 = \{\emptyset\}$. Let $I_\emptyset = [0,1]$, and if the interval $I_{i_1\dots i_n} = [u,v]$ is already defined then let

$$I_{i_1\dots i_n 0} = [u, u+\gamma_{n+1}] \quad \text{and} \quad I_{i_1\dots i_n 1} = [v-\gamma_{n+1}, v].$$

Let

$$A = \bigcap_{n=0}^{\infty} \bigcup_{(i_1,\dots,i_n)\in\{0,1\}^n} I_{i_1\dots i_n}.$$

Assume that $f:[0,1] \to \mathbb{R}$ is a function satisfying (4.5). Now we prove that $\operatorname{Var}^\beta(f|_A) < \infty$. As $\operatorname{diam} I_{i_1\dots i_n} = \gamma_n$, the definition of γ_n and (4.5) imply that for all $n \in \mathbb{N}$ and $(i_1,\dots,i_n) \in \{0,1\}^n$ we have

$$\left(\operatorname{diam} f(I_{i_1\dots i_n})\right)^\beta \le \left(c\gamma_n^\alpha \log \gamma_n^{-1}\right)^\beta \le c_{\alpha,\beta} 2^{-n}(n+1)^{-2}, \tag{4.6}$$

where $c_{\alpha,\beta} \in (0,\infty)$ is a constant depending on α,β and c only. For all $x, y \in A$ let $n(x,y)$ be the maximal number n such that $x, y \in I_{i_1\dots i_n}$ for some $(i_1,\dots,i_n) \in \{0,1\}^n$. If $\{x_i\}_{i=0}^k$ is a monotone sequence in A and $n \in \mathbb{N}$, then the number of $i \in \{1,\dots,k\}$ such that $n(x_{i-1},x_i) = n$ is at most 2^n. Therefore (4.6) implies that

$$\operatorname{Var}^\beta(f|_A) \le \sum_{n=0}^{\infty} 2^n\left(c_{\alpha,\beta}2^{-n}(n+1)^{-2}\right) = \sum_{n=1}^{\infty} c_{\alpha,\beta} n^{-2} < \infty.$$

Finally, we prove that $\dim A = \alpha\beta$. The upper bound $\dim A \le \alpha\beta$ is obvious, thus we show only the lower bound. In the construction of A each $(n-1)$st-level interval $I_{i_1\dots i_{n-1}}$ contains $m_n = 2$ nth-level intervals $I_{i_1\dots i_{n-1}i}$, which are separated by gaps of $\varepsilon_n = \gamma_{n-1} - 2\gamma_n$. The definition of γ_n yields that $0 < \varepsilon_{n+1} < \varepsilon_n$ for all $n \in \mathbb{N}^+$ and $\varepsilon_n = 2^{-n/(\alpha\beta)+o(n)}$. Applying [4, Example 4.6] we obtain that:

$$\dim A \ge \liminf_{n\to\infty} \frac{\log(m_1\cdots m_{n-1})}{-\log(m_n\varepsilon_n)} = \alpha\beta,$$

and the proof is complete. □

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References

[1] O. Angel, R. Balka, Y. Peres, Increasing subsequences of random walks, preprint, arXiv:1407.2860.
[2] T. Antunović, K. Burdzy, Y. Peres, J. Ruscher, Isolated zeros for Brownian motion with variable drift, Electron. J. Probab. 16 (65) (2011) 1793–1814.
[3] M. Elekes, Hausdorff measures of different dimensions are isomorphic under the Continuum Hypothesis, Real Anal. Exch. 30 (2) (2004) 605–616.
[4] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, second edition, John Wiley & Sons, 2003.
[5] J. Hawkes, On the Hausdorff dimension of the intersection of the range of a stable process with a Borel set, Z. Wahrscheinlichkeit 19 (1971) 90–102.
[6] J.-P. Kahane, Y. Katznelson, Restrictions of continuous functions, Isr. J. Math. 174 (2009) 269–284.
[7] R. Kaufman, Une propriété métrique du mouvement brownien, C. R. Acad. Sci. Paris 268 (1969) 727–728.
[8] R. Kaufman, Measures of Hausdorff-type, and Brownian motion, Mathematika 19 (1972) 115–119.
[9] P. Lévy, Théorie de l'addition des variables aléatoires, Gauthier–Villars, Paris, 1937.
[10] A. Máthé, Measurable functions are of bounded variation on a set of Hausdorff dimension $\frac{1}{2}$, Bull. Lond. Math. Soc. 45 (2013) 580–594.
[11] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, UK, 1995.
[12] P. Mörters, Y. Peres, Brownian Motion, with an appendix by Oded Schramm and Wendelin Werner, Cambridge University Press, Cambridge, UK, 2010.