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The obstruction to the existence of a loopless star product

Thomas Willwacher [1]

Department of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zürich, Switzerland

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ABSTRACT

We show that there is an obstruction to the existence of a star product defined by Kontsevich graphs without directed cycles.

RÉSUMÉ

Nous montrons qu'il y a une obstruction à l'éxistence d'une produit étoile défini par les graphes de Kontsevich sans cycle orienté.

1. Introduction

The theory of deformation quantization [1] studies the existence and uniqueness of $\star$-products, i.e., of associative $\mathbb{R}[\hbar]$ linear products on $C^\infty(M)[[\hbar]]$ (for M a smooth manifold) having the form

$$f \star g = fg + \hbar m_1(f,g) + \hbar^2 m_2(f,g) + \cdots$$

where the m_j are bidifferential operators. We are interested only in the case when $M = \mathbb{R}^d$, d very large, and of m_j determined from a Poisson bivector field π by universal formulas, by which we mean Kontsevich graphs. See [3] for the definition of those graphs. In other words, we may replace a star product with a formal series of Kontsevich graphs

$$a = a_0 + a_1 + a_2 + \cdots$$

for our purposes, where a_0 is fixed to be the graph

$a_0 =$.

The space of formal series of Kontsevich graphs forms a graded Lie algebra and the condition of associativity of the star product translates into the Maurer–Cartan equation

$$[a,a] = 0. \tag{1}$$

E-mail address: thomas.willwacher@math.uzh.ch.

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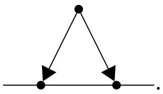
M. Kontsevich gave explicit formulas for a series of Kontsevich graphs satisfying (1), which we denote by

$$a^K = a_0^K + a_1^K + a_2^K + a_3^K + a_4^K + \cdots$$

where a_j^K is a linear combination of graphs with j type I vertices. In particular $a_0^K = a_0$ and

$$a_1^K = \frac{1}{2}$$  .

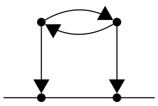
For the detailed construction of a^K, we refer the reader to the original paper [3]. Kontsevich's formal series gives rise to a star product on $\mathbb{R}^d$, for finite d, but cannot be used to construct a star product in infinite dimensions, due to the existence of directed loops in graphs occurring in a_K. It was asked whether the formula can be modified in such a manner that no graphs with directed loops occur. This question has been studied by S. Merkulov [4], who showed that a loopless star product cannot exist in the graded setting and by B. Shoikhet [6], who showed that a loopless formality morphism cannot exist.

This note contains a small calculation showing that a loopless star product does not exist in the non-graded situation either. The result is independently shown by G. Dito in the recent preprint [2] as well, using an earlier explicit calculation by Penkava and Vanhaecke [5]. The benefit of our calculation is that it is a little shorter, while not using the result of [5].

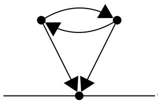
2. Star product

As indicated above, we call a universal star product a star product given by a formal sum of Kontsevich graphs, with two type-II vertices and with all type-I vertices having exactly two outgoing edges (see [3] for the notation). The type-I vertices formally represent copies of a Poisson bivector field π. We furthermore identify two linear combinations of graphs if they can be transformed to each other using only the (graphical version of the) Maurer–Cartan equation $[\pi, \pi] = 0$ for a Poisson bivector field.

We shall try to construct a universal star product using only graphs without oriented loops. Note that for all star products a_0 and a_1 are fixed by definition, there is no choice. a_2 is the same for all loopless star products and in this case uniquely determined by the MC equation. However, note that in the Kontsevich product a_2^K contains a loop graph, namely



with nonzero weight 4α, say. It can be removed by performing a gauge transformation using the graph

$L =$  .

So we define

$$a := \exp\big(2\alpha[L, \cdot]\big)a^K = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots.$$

Then $a_0 = a_0^K$, $a_1 = a_1^K$ and a_2 contains no graphs with directed cycles.

Now suppose that our other (desirably loopless) star product reads

$$b = a_0 + a_1 + a_2 + (a_3 + b_3) + (a_4 + b_4) + \cdots.$$

Then the Maurer–Cartan equation $[b, b] = 0$ implies in particular:

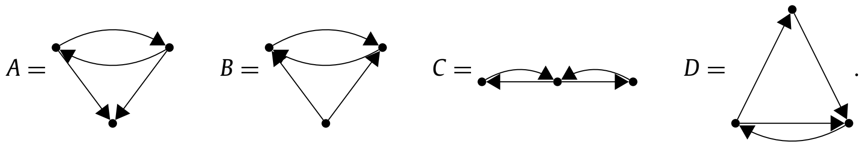
$$0 = [a_0, b_3]$$
$$0 = [a_1, b_3] + [a_0, b_4].$$

These imply:

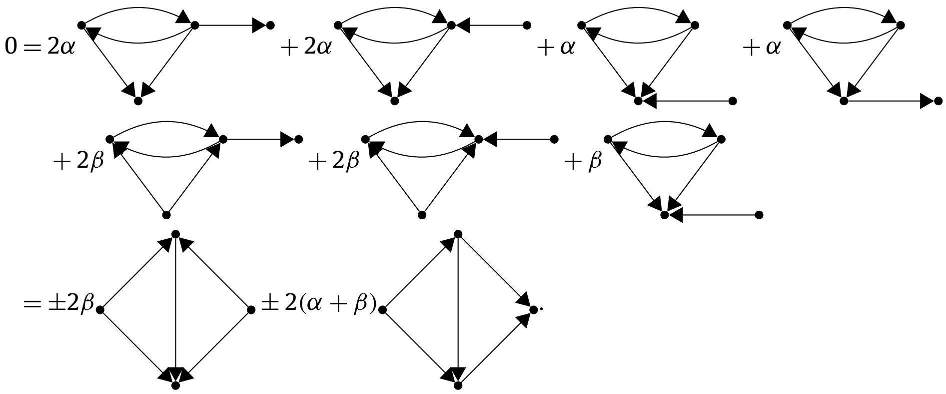
$$0 = [a_0, b_3] \tag{2}$$
$$0 = [a_1, b_3] \quad \text{modulo the image of } [a_0, \cdot]. \tag{3}$$

Claim: For any solution b_3 to (2) and (3), the sum $a_3 + b_3$ contains graphs with directed cycles.

Solvability is unchanged if we add some expression $[a_0, X]$ to b_3, since $[a_0, a_1] = 0$. Hence we can assume that b_3 is in fact in the image of the (graphical version of the) Hochschild–Kostant–Rosenberg map. In other words, it is composed of graphs for which both type II vertices have valence 1, and which are antisymmetric under interchange of these vertices. So we may just omit the type II vertices and adjacent edges in drawings, they can be uniquely recovered. We have the following 4 candidate graphs:



Of these, graphs A, C and D have weight 0 in the Kontsevich star product a^K. Graph B has non-zero weight $-\beta$, say (see [7,8] for a computation of β). The graph A attains non-zero weight $-\alpha$ in a_3 due to the gauge transformation, while the weights of the other graphs remain unchanged. It follows that in order for $a_3 + b_3$ to be loopless, b_3 must be a linear combination of graphs A and B, namely $b_3 = \alpha A + \beta B$. Let us insert this b_3 into Eq. (3). We obtain:



For the last line, one must use that $[\pi, \pi] = 0$. One checks that this combination of graphs non-zero, irrespective of the values of the non-zero numbers α, β.[2] Hence we arrive at a contradiction and the claim is shown. Note again that all vertices should be understood as having two outgoing edges. If they have less in the drawing, one must add other outgoing edges to type-II vertices. As a direct corollary of the claim, we arrive at our main result:

Theorem 1. *Any formal series of Kontsevich graphs defining a universal star product necessarily contains a graph with oriented cycles.*

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[2] This is Shoikhet's obstruction, cf. [6].