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Complex analysis

Faber polynomial coefficient estimates for analytic bi-close-to-convex functions

Estimation des coefficients des fonctions analytiques bi-presque convexes à l'aide des polynômes de Faber

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ABSTRACT

Using the Faber polynomials, we obtain coefficient expansions for analytic bi-close-to-convex functions and determine coefficient estimates for such functions. We also demonstrate the unpredictable behavior of the early coefficients of subclasses of bi-univalent functions. A function is said to be bi-univalent in a domain if both the function and its inverse map are univalent there.

RÉSUMÉ

Nous exprimons les coefficients des développements de fonctions analytiques bi-presque convexes en utilisant les polynômes de Faber, et nous en déduisons des estimations de ces coefficients. Une fonction est dite bi-univalente dans un domaine si elle et son inverse sont univalentes dans ce domaine. Nous montrons également le comportement imprévisible des premiers coefficients pour des sous-classes de fonctions bi-univalentes.

1. Introduction

Let $\mathcal{A}$ denote the family of functions analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$ and let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$ and normalized by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For α; $0 \leqslant \alpha < 1$, we let $\mathcal{S}^*(\alpha)$ denote the class of function $g \in \mathcal{S}$ that are starlike of order α in $\mathbb{D}$, that is, $\mathrm{Re}[zg'(z)/g(z)] > \alpha$ in $\mathbb{D}$ and $C(\alpha)$ denote the class of functions $f \in \mathcal{S}$ that are close-to-convex of order α in $\mathbb{D}$, that is, if there exists a function $g \in \mathcal{S}^*(0)$ so that $\mathrm{Re}[zf'(z)/g(z)] > \alpha$ in $\mathbb{D}$ (e.g. see [8] or [12]). We note that $\mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ and that $|a_n| \leqslant n$ for $f \in \mathcal{S}$ by de Branges' Theorem [7], also known as the Bieberbach Conjecture (e.g., see [7] or [8]).

If $F = f^{-1}$ is the inverse of a function $f \in S$, then F has a Maclaurin series expansion in some disk about the origin [8]. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse map $F = f^{-1}$ are univalent in $\mathbb{D}$. By the same

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token, a function $f \in \mathcal{A}$ is said to be bi-close-to-convex of order α in $\mathbb{D}$ if both f and its inverse map $F = f^{-1}$ are close-to-convex of order α in $\mathbb{D}$. The class of bi-univalent analytic functions was first introduced and studied by Lewin [13], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [5] improved Lewin's result to $|a_2| \leqslant \sqrt{2}$ and later Netanyahu [14] proved that $|a_2| \leqslant 4/3$. Brannan and Taha [6] and Taha [15] also investigated certain subclasses of bi-univalent functions and found estimates for their first couple of coefficients. Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behavior of the coefficients of the function f and its inverse $F = f^{-1}$ unpredictable. Here, in this paper, we use the Faber polynomial expansions to determine estimates for the general coefficients $|a_n|$ of bi-close-to-convex functions under certain gap series condition. We then demonstrate the unpredictability of the coefficient behavior of bi-starlike functions and provide an example of a bi-close-to-convex function. The bi-close-to-convex functions considered in this paper are the largest subclass of bi-univalent functions thus far investigated and no coefficient estimates for the functions in this class have yet appeared in the literature.

2. Main results

Using the Faber polynomial expansion of functions $f \in \mathcal{S}$ of the form (1.1), the coefficients of its inverse map $F = f^{-1}$ may be expressed as (e.g. see [3, Eq. (1.33), page 185]):

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n = w + \sum_{n=2}^{\infty} A_n w^n, \tag{2.1}$$

$$\begin{aligned} K_{n-1}^{-n} = {} & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \big[a_5 + (-n+2) a_3^2\big] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \big[a_6 + (-2n+5) a_3 a_4\big] + \sum_{j \geqslant 7} a_2^{n-j} V_j, \end{aligned}$$

where V_j is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see [4] and [3]). In particular, the first few terms of K_{n-1}^{-n} are $K_1^{-2} = -2a_2$, $K_2^{-3} = 3(2a_2^2 - a_3)$ and $K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4)$. In general, an expansion of $K_{n-1}^{p}(a_2, a_3, \ldots, a_n)$ is given by:

$$K_{n-1}^{p} = p a_n + \frac{p(p-1)}{2} D_{n-1}^{2} + \frac{p!}{(p-3)!3!} D_{n-1}^{3} + \cdots + \frac{p!}{(p-n+1)!(n-1)!} D_{n-1}^{n-1},$$

where $D_{n-1}^{p} = D_{n-1}^{p}(a_2, a_3, \ldots, a_n)$,

$$D_{n-1}^{m}(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!} \quad \text{for } m \leqslant n,$$

and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_{n-1}$ satisfying $\mu_1 + \mu_2 + \cdots + \mu_{n-1} = m$ and $\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1$. Evidently: $D_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$ (see [1], [2] or [16]).

The Faber polynomials introduced by Faber [9] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [10] and [11] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions. As a result, we are able to state and prove the following.

Theorem 2.1. *For $0 \leqslant \alpha < 1$ let the function $f \in \mathcal{S}$ be bi-close-to-convex of order α in $\mathbb{D}$. If $a_k = 0$; $2 \leqslant k \leqslant n-1$, then:*

$$|a_n| \leqslant 1 + \frac{2(1-\alpha)}{n}.$$

Proof. First let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be close-to-convex of order α in $\mathbb{D}$. Therefore, there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(0)$ so that $\mathrm{Re}[zf'(z)/g(z)] > \alpha$ in $\mathbb{D}$. The Faber polynomial expansion for $zf'(z)/g(z)$ is:

$$\frac{zf'(z)}{g(z)} = 1 + \sum_{n=2}^{\infty} \left[(na_n - b_n) + \sum_{\ell=1}^{n-2} K_{\ell}^{-1}(b_2, b_3, \ldots, b_{\ell+1}) \big((n-\ell) a_{n-\ell} - b_{n-\ell} \big) \right] z^{n-1}. \tag{2.2}$$

For the inverse map $F = f^{-1}$ to be close-to-convex of order α in $\mathbb{D}$, there exists a function $G(w) = w + \sum_{n=2}^{\infty} B_n w^n \in \mathcal{S}^*(0)$ so that $\operatorname{Re}[wF'(w)/G(w)] > \alpha$ in $\mathbb{D}$. According to (2.2), the Faber polynomial expansion of the inverse map $F = f^{-1}$ is $F(w) = w + \sum_{n=2}^{\infty} A_n w^n$. Thus the Faber polynomial expansion of $wF'(w)/G(w)$ is given by:

$$\frac{wF'(w)}{G(w)} = 1 + \sum_{n=2}^{\infty} \left[(nA_n - B_n) + \sum_{\ell=1}^{n-2} K_\ell^{-1}(B_2, B_3, \ldots, B_{\ell+1})\big((n-\ell)A_{n-\ell} - B_{n-\ell}\big) \right] w^{n-1}. \tag{2.3}$$

On the other hand, since $\operatorname{Re}[zf'(z)/g(z)] > \alpha$ in $\mathbb{D}$, there exists a positive real part function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$ so that:

$$\frac{zf'(z)}{g(z)} = \alpha + (1-\alpha)p(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n. \tag{2.4}$$

Similarly, for $\operatorname{Re}[wF'(w)/G(w)] > \alpha$ in $\mathbb{D}$, there exists a positive real part function $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A}$ so that:

$$\frac{wF'(w)}{G(w)} = \alpha + (1-\alpha)q(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n. \tag{2.5}$$

Note that, by the Caratheodory lemma (e.g., [8]), $|c_n| \leqslant 2$ and $|d_n| \leqslant 2$.

Comparing the coefficients of Eqs. (2.2) and (2.4), for any $n \geqslant 2$, yields:

$$(na_n - b_n) + \sum_{\ell=1}^{n-2} K_\ell^{-1}(b_2, b_3, \ldots, b_{\ell+1})\big((n-\ell)a_{n-\ell} - b_{n-\ell}\big) = (1-\alpha)c_{n-1}. \tag{2.6}$$

Similarly, from (2.3) and (2.5), we obtain:

$$(nA_n - B_n) + \sum_{\ell=1}^{n-2} K_\ell^{-1}(B_2, B_3, \ldots, B_{\ell+1})\big((n-\ell)A_{n-\ell} - B_{n-\ell}\big) = (1-\alpha)d_{n-1}. \tag{2.7}$$

For the special case $n = 2$, Eqs. (2.6) and (2.7), respectively, yield $2a_2 - b_2 = (1-\alpha)c_1$ and $-(2a_2 + B_2) = (1-\alpha)d_1$. Solving for a_2 and taking the absolute values, we obtain $|a_2| \leqslant 2 - \alpha$. But under the assumption $a_k = 0$, $2 \leqslant k \leqslant n-1$, Eqs. (2.6) and (2.7), respectively, yield:

$$na_n - b_n = (1-\alpha)c_{n-1} \tag{2.8}$$

and

$$na_n - B_n = (1-\alpha)d_{n-1}. \tag{2.9}$$

Solving either of Eqs. (2.8) or (2.9) for a_n and taking the absolute values, we obtain $|a_n| \leqslant 1 + 2(1-\alpha)/n$ upon noticing that $|b_n| \leqslant n$ and $|B_n| \leqslant n$. □

As a special case to Theorem 2.1, we have the following corollary that clearly demonstrates the unpredictability of the coefficient behavior of subclasses of bi-univalent functions.

Corollary 2.2. *For $0 \leqslant \alpha < 1$ let $f \in \mathcal{S}^*(\alpha)$ and $F = f^{-1} \in \mathcal{S}^*(\alpha)$. Then:*

(i) $|a_2| \leqslant \begin{cases} \sqrt{2(1-\alpha)}, & 0 \leqslant \alpha < \frac{1}{2}; \\ 2(1-\alpha), & \frac{1}{2} \leqslant \alpha < 1. \end{cases}$

(ii) $|a_3| \leqslant \begin{cases} 2(1-\alpha), & 0 \leqslant \alpha < \frac{1}{2}; \\ (1-\alpha)(3-2\alpha), & \frac{1}{2} \leqslant \alpha < 1. \end{cases}$

Proof. We notice that, for the bi-starlike case, the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ will be the same as the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the proof of Theorem 2.1, that is, $b_n = a_n$ there.

For $n = 2$, Eqs. (2.6) and (2.7), respectively, yield $a_2 = (1-\alpha)c_1$ and $-a_2 = (1-\alpha)d_1$. Taking the absolute values of either of these two equations gives $|a_2| \leqslant 2(1-\alpha)$.

For $n = 3$, Eqs. (2.6) and (2.7), respectively, yield:

$$2a_3 - a_2^2 = (1-\alpha)c_2, \tag{2.10}$$

and

$$-2a_3 + 3a_2^2 = (1-\alpha)d_2. \tag{2.11}$$

Adding the above two equations and solving for $|a_2|$, we obtain:

$$|a_2| = \sqrt{\frac{(1-\alpha)|c_2 + d_2|}{2}} \leqslant \sqrt{2(1-\alpha)}.$$

Consequently, we obtain the estimate (i) upon noting that $\sqrt{2(1-\alpha)} < 2(1-\alpha)$ if $\alpha < 1/2$.

Multiplying Eq. (2.10) by 3 and adding it to (2.11), we obtain $4a_3 = (1-\alpha)(3c_2 + d_2)$. Then, solving for $|a_3|$ yields:

$$|a_3| = \frac{(1-\alpha)|3c_2 + d_2|}{4} \leqslant 2(1-\alpha).$$

Substituting $a_2 = (1-\alpha)c_1$ in (2.10) gives $2a_3 = (1-\alpha)[c_2 + (1-\alpha)c_1^2]$. Therefore,

$$|a_3| = \frac{(1-\alpha)|c_2 + (1-\alpha)c_1^2|}{2} \leqslant \frac{(1-\alpha)(|c_2| + (1-\alpha)|c_1|^2)}{2} \leqslant (1-\alpha)(3-2\alpha).$$

The estimates in part (ii) follow upon noting that $2 < 3 - 2\alpha$ if $\alpha < 1/2$. □

In the following, we give an example of a bi-close-to-convex function.

Example 2.3. For $n \geqslant 3$, we will show that $f(z) = z + \frac{1-\alpha}{n-1}z^n$ is bi-close-to-convex of order α; $0 \leqslant \alpha < 1$ in $\mathbb{D}$. For the function $g(z) = z - \frac{1-\alpha}{n-\alpha}z^n$ starlike in $\mathbb{D}$, we have:

$$\frac{zf'(z)}{g(z)} = \frac{1 + \frac{n(1-\alpha)}{n-1}z^{n-1}}{1 - \frac{1-\alpha}{n-\alpha}z^{n-1}} = 1 + \sum_{k=1}^{\infty}\left(\frac{(1-\alpha)^k}{(n-\alpha)^k} + \frac{n(1-\alpha)^k}{(n-1)(n-\alpha)^{k-1}}\right)z^{(n-1)k}.$$

Therefore,

$$\frac{\frac{zf'(z)}{g(z)} - \alpha}{1-\alpha} = 1 + \sum_{k=1}^{\infty}\frac{n^2 + (1-\alpha)n - 1}{(n-1)(n-\alpha)}\left(\frac{1-\alpha}{n-\alpha}\right)^{k-1} z^{(n-1)k} = \frac{M_0}{2} + \sum_{k=1}^{\infty} M_k z^{(n-1)k}.$$

We observe that M_k is a convex null sequence since $\lim_{k\to\infty} M_k = 0$ and:

$$M_0 - M_1 \geqslant M_1 - M_2 \geqslant \cdots \geqslant M_k - M_{k+1} \geqslant \cdots \geqslant 0.$$

Therefore $\mathrm{Re}(\frac{zf'(z)}{g(z)} - \alpha) > 0$ in $\mathbb{D}$.

For the inverse map $F = f^{-1}$, we have $F(w) = w - \frac{1-\alpha}{n-1}w^n$ and then choose the function $G(w) = w + \frac{1-\alpha}{n-\alpha}w^n$ which is starlike in $\mathbb{D}$. Consequently, we have:

$$\frac{\frac{wF'(w)}{G(w)} - \alpha}{1-\alpha} = \frac{2}{2} + \sum_{k=1}^{\infty}(-1)^k\frac{n^2 + (1-\alpha)n - 1}{(n-1)(n-\alpha)}\left(\frac{1-\alpha}{n-\alpha}\right)^{k-1} w^{(n-1)k}.$$

Obviously, $\mathrm{Re}(\frac{wF'(w)}{G(w)} - \alpha) > 0$ since its coefficients are dominated by the convex null sequence M_k.

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