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Bach-flat Lie groups in dimension 4

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ABSTRACT

We establish the existence of solvable Lie groups of dimension 4 and left-invariant Riemannian metrics with zero Bach tensor which are neither conformally Einstein nor half conformally flat.

RÉSUMÉ

Nous montrons l'existence de groupes de Lie résolubles de dimension 4 et de métriques riemanniennes invariantes à gauche, dont le tenseur de Bach est nul et qui ne sont ni conformément Einstein, ni semi-conformément plates.

1. Introduction

Let (M, g) be a 4-dimensional Riemannian manifold, let ∇ denote its Levi-Civita connection, R its Riemann tensor and W its Weyl tensor. The latter depends only on the conformal class $[g]$ of the metric g and decomposes as $W = W_+ + W_-$. Another conformal invariant, the Bach tensor B, is the irreducible component of $\nabla\nabla R$ that, if M is compact, corresponds to the gradient of the Lagrangian:

$$g \mapsto \int_M \|W[g]\|^2 \,\mathrm{d}\upsilon_g,$$

in which one may, for topological reasons, replace W by W_+ or W_-. The tensor B can be regarded as trace-free symmetric bilinear form, and vanishes if M is self-dual ($W_- = 0$) or anti-self-dual ($W_+ = 0$), in other words if M is half conformally flat. It also vanishes whenever $[g]$ has an Einstein representative, and this aspect was studied by Derdziński [7] (see also [2,5]). Metrics with zero Bach tensor therefore form a natural class in which to generalize results on Einstein metrics and curvature flow [1,16,6].

There are few known examples of 'non-trivial' Bach-flat metrics, meaning ones with zero Bach tensor satisfying neither the Weyl nor the Einstein condition above. A construction is outlined by Schmidt [15] and an explicit Lorentzian one is given by Nurowski and Plebański [14]. In the light of the classification of Einstein, hyperhermitian and self-dual metrics on 4-dimensional Lie groups by Jensen [9] and others [4,3,8], it is natural to ask whether there exists a Lie group with a non-trivial left-invariant Bach-flat metric. This note answers the question by providing two examples. The Lie groups are not unimodular and (predictably) the metrics do not pass to compact quotients. Nonetheless, they have a privileged status in the study of the curvature of left-invariant metrics in the spirit of Milnor's paper [13].

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2. A family of metric Lie algebras

Consider the Lie algebra $\mathfrak{g}_{\alpha,\beta}$ defined by a basis (e_i) of $\mathbb{R}^4$ with non-zero brackets

$$[e_2,e_1]=\alpha e_2,\qquad [e_3,e_1]=\beta e_3,\qquad [e_4,e_1]=(\alpha+\beta)e_4,\qquad [e_3,e_2]=e_4,$$

where α and β are non-zero real numbers. This gives a family of solvable, non-nilpotent Lie algebras, which are unimodular if and only if $\alpha+\beta=0$. If (e^i) is the dual basis of (e_i), then one uses the formula $\mathrm{d}e^i(e_j,e_k)=-e^i([e_j,e_k])$ to encode the structure constants into the differential system

$$\begin{cases} \mathrm{d}e^1=0, \\ \mathrm{d}e^2=\alpha e^1\wedge e^2, \\ \mathrm{d}e^3=\beta e^1\wedge e^3, \\ \mathrm{d}e^4=(\alpha+\beta)e^1\wedge e^4+e^2\wedge e^3. \end{cases} \tag{1}$$

Once one passes to an associated Lie group, d may be regarded as the exterior derivative on the space of left-invariant 1-forms.

Proposition 2.1. *If $\alpha\neq\beta$, there exist Lie algebra isomorphisms*

- $P_\lambda:\mathfrak{g}_{\alpha,\beta}\to\mathfrak{g}_{\lambda\alpha,\lambda\beta}$ *for each* $\lambda\neq 0$;
- $Q:\mathfrak{g}_{\alpha,\beta}\to\mathfrak{g}_{\beta,\alpha}$.

Any isomorphism $\mathfrak{g}_{\alpha,\beta}\to\mathfrak{g}_{\alpha',\beta'}$ is generated in this way, so (α',β') equals either $(\lambda\alpha,\lambda\beta)$ or $(\lambda\beta,\lambda\alpha)$ for some $\lambda\neq 0$. If $\alpha=\beta$, then necessarily $\lambda=\pm 1$.

In terms of a basis (e^i) defining d, the first isomorphism is given by $P_\lambda(e^1)=e^1/\lambda$ and $P_\lambda(e^i)=e^i$ for $i>1$. The second, Q, merely swaps e^2 and e^3. To prove the converse, one can characterize the isomorphism class by the set of 1-forms satisfying $\omega\wedge\mathrm{d}\omega=0$ where $\omega=\sum_{i=1}^4 a_i e^i$.

Endow $\mathfrak{g}_{\alpha,\beta}$ with the inner product for which (e^i) is (dual to) an orthonormal basis. Two Lie algebras equipped with inner products can be called *isometric* if there exists a Lie algebra isomorphism between them that is an isometry. In particular, $\mathfrak{g}_{\alpha,\beta}$ is isometric to both $\mathfrak{g}_{\beta,\alpha}$ and $\mathfrak{g}_{-\alpha,-\beta}$.

3. Curvature calculations

Fix non-zero constants α,β, and let M be a Lie group with Lie algebra $\mathfrak{g}_{\alpha,\beta}$. We endow M with the left-invariant Riemannian metric $g=\sum_{i=1}^4 e^{ii}$, where $e^{ii}=e^i\otimes e^i$. Let R denote its Riemann tensor, ρ the Ricci tensor, and τ the scalar curvature, so that $\rho_{ij}=\sum_{k=1}^4 R_{ikj}^k=\sum_{k,l=1}^4 R_{kilj}g^{kl}$ and $\tau=\sum_{i=1}^4\rho_{ii}$. From the Cartan structure equations, we obtain:

$$\rho=-\big(\alpha^2+\beta^2+\alpha\beta\big)e^{11}-\left(\alpha^2+\alpha\beta+\frac{1}{4}\right)e^{22}-\left(\beta^2+\alpha\beta+\frac{1}{4}\right)e^{33}-\left((\alpha+\beta)^2-\frac{1}{4}\right)e^{44}.$$

Observe that g is Einstein if and only if $\alpha=\beta=\pm\frac{1}{2}$; this is the symmetric metric on the complex hyperbolic plane $\mathbb{C}H^2$ [9].

The Bach tensor B can be defined by:

$$B_{ij}=\sum_{p=1}^4\nabla_p\nabla_j\rho_{ip}-\frac{1}{2}\sum_{p=1}^4\nabla_p\nabla_p\rho_{ij}+\frac{1}{3}\tau\rho_{ij}-\sum_{p=1}^4\rho_{pi}\rho_{pj}+\frac{1}{12}\left[3\sum_{r,s=1}^4(\rho_{rs})^2-\tau^2\right]\delta_{ij}.$$

It turns out to be diagonal and, since $\sum_{i=1}^4 B_{ii}=0$, we need only record:

$$\begin{aligned} B_{11}&=\frac{1}{6}-\frac{1}{6}\alpha^2+\frac{2}{3}\alpha^3\beta-\frac{2}{3}\alpha^2\beta^2-\frac{1}{2}\alpha\beta+\frac{2}{3}\alpha\beta^3-\frac{1}{6}\beta^2, \\ B_{22}&=\frac{5}{6}\alpha^2+\frac{1}{2}\beta^2+\frac{2}{3}\alpha^3\beta-2\alpha\beta^3+\frac{7}{6}\alpha\beta-\frac{2}{3}\alpha^2\beta^2-\frac{1}{2}, \\ B_{33}&=\frac{1}{2}\alpha^2+\frac{5}{6}\beta^2+\frac{7}{6}\alpha\beta-2\alpha^3\beta+\frac{2}{3}\alpha\beta^3-\frac{2}{3}\alpha^2\beta^2-\frac{1}{2}. \end{aligned}$$

It is incredibly easy to resolve the system $B=0$, since it yields $(\alpha^2-\beta^2)(1+8\alpha\beta)=0$. If the second factor is zero then α,β are roots of $8x^4-7x^2+\frac{1}{8}=0$. Let us denote these roots by $\pm r_1,\pm r_2$ where:

$$r_1 = \frac{1}{4}\sqrt{7-3\sqrt{5}} = \frac{1}{8}(3\sqrt{2}-\sqrt{10}), \qquad r_2 = -\frac{1}{4}\sqrt{7+3\sqrt{5}} = -\frac{1}{8}(3\sqrt{2}+\sqrt{10}).$$

We obtain eight solutions which, by Proposition 2.1, fall into three essentially distinct classes:

$$\text{(i)} \quad (\alpha,\beta) = \pm(1,1), \qquad \text{(ii)} \quad (\alpha,\beta) = \pm\left(\frac{1}{2},\frac{1}{2}\right), \qquad \text{(iii)} \quad (\alpha,\beta) = \pm(r_1,r_2) \text{ or } \pm(r_2,r_1). \tag{2}$$

Solutions (i) and (ii) correspond to those of [8]. The first is hyperhermitian (so $W_+ = 0$) and the second is the Einstein metric on $\mathbb{C}H^2$, so the vanishing of their respective Bach tensor is already known. Our next goal is to show that (iii) is a non-trivial Bach-flat metric.

We know that (iii) is not half conformally flat, by [8], and this is verified by direct computation of $W_\pm$. From a result of Listing ([11, Proposition 1]; see also [10]), a necessary condition for a 4-dimensional manifold to be locally conformally Einstein is the existence of a non-zero vector field $T = \sum_{k=1}^4 x_k e_k$ satisfying $(\operatorname{div}_4 W)(X,Y,Z) = W(X,Y,Z,T)$, where:

$$(\operatorname{div}_4 W)(e_h,e_k,e_p) = -\sum_{i,q=1}^{4}\left[\omega_h^q(e_i)W_{qkpi} + \omega_k^q(e_i)W_{hqpi} + \omega_p^q(e_i)W_{hkqi} + \omega_i^q(e_i)W_{hkpq}\right],$$

and $\omega_j^i(e_k) = e^i(\nabla_{e_k} e_j)$. In case (iii), we discover a contradiction by examining $(\operatorname{div}_4 W)(e_1,e_2,e_j)$ for $j=1$ and $j=2$. Thus,

Theorem 3.1. *Let G be a simply-connected 4-dimensional Lie group associated with the solvable Lie algebra $\mathfrak{g}_{r_1,r_2}$ defined by (1), and let $h = \sum_{i=1}^4 e^{ii}$. The Riemannian metric h is non-trivially Bach flat.*

The Ricci tensor of h is diagonal relative to the basis (e^i) with entries $-\frac{3}{2}$, $\frac{3}{8}(-3+\sqrt{5})$, $-\frac{3}{8}(3+\sqrt{5})$, $-\frac{3}{4}$. The tensor $8W_\pm$ has eigenvalues $2\pm(3\sqrt{2}-\sqrt{10})$, $2\mp(3\sqrt{2}+\sqrt{10})$, $-4\pm2\sqrt{10}$.

4. A 2-step solvable example

In order to undertake a more general study of possible left-invariant Bach-flat metrics, one has to work with an orthonormal basis of 1-forms satisfying more complicated differential relations. The solvable case can be tackled by a case-by-case analysis following Jensen's work on the classification of left-invariant Einstein metrics [9]. The examples (2) lie in the class of Lie algebras for which $\dim\mathfrak{g}' = 3$ and $\dim\mathfrak{g}'' = 1$ (where $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}'' = [\mathfrak{g}',\mathfrak{g}']$ are the first two terms of the derived series). Let G be a 4-dimensional Lie group with such a Lie algebra, admitting a left-invariant Bach-flat metric g. One can show that there are three possibilities: (i) g is conformally Einstein, (ii) one of W_+, W_- is zero, or (iii) $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{r_1,r_2}$ and g is homothetic to h.

By rotating in the plane $\langle e^2, e^3\rangle$ and applying an overall re-scaling, we may convert $\mathfrak{g}_{r_1,r_2}$ into a form that simplifies the coefficients in $\mathbb{Q}(\sqrt{2},\sqrt{5})$. Namely, $\mathrm{d}e^1 = 0$ and

$$\mathrm{d}e^2 = e^1\wedge e^3, \qquad \mathrm{d}e^3 = e^1\wedge e^2 + \sqrt{5}e^1\wedge e^3, \qquad \mathrm{d}e^4 = \sqrt{5}e^1\wedge e^4 + 2\sqrt{2}e^2\wedge e^3.$$

This therefore describes a Bach-flat metric homothetic to h. In view of Theorem 3.1, it is natural to ask whether there are solutions in other number fields. In fact, there exists a 2-step example (meaning that $\mathfrak{g}''$ vanishes):

Theorem 4.1. *There exists a solvable Lie algebra $\mathfrak{g}$ defined over $\mathbb{Q}(\sqrt{2},\sqrt{3})$ with $\dim\mathfrak{g}' = 2$ and $\mathfrak{g}'' = \{0\}$, whose associated Lie group admits a non-trivial Bach-flat metric.*

The metric Lie algebra in question is given by $\mathrm{d}e^1 = 0 = \mathrm{d}e^2$ and

$$\mathrm{d}e^3 = \sqrt{2}e^1\wedge e^3 + e^2\wedge e^3 - (1/\sqrt{3})e^2\wedge e^4, \qquad \mathrm{d}e^4 = \sqrt{2}e^1\wedge e^4 - e^2\wedge e^4 + (1/\sqrt{3})e^2\wedge e^3.$$

The examples of Theorems 3.1 and 4.1 are obviously very special. A study of other isomorphism classes described in [12] leads us to predict that there are no continuous families of solvable Lie algebras admitting non-trivial left-invariant Bach-flat metrics, in contrast to the self-dual case [8]. We can assert that there are no new solutions on unimodular Lie groups, since the unimodular condition simplifies the equations, in particular in the nilpotent cases and in the reductive cases $\mathfrak{gl}(2,\mathbb{R})$, $\mathfrak{u}(2)$.

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