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A global attractor for a $p(x)$-Laplacian inclusion

Un attracteur global d'une inclusion avec $p(x)$-Laplacien

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ABSTRACT

In this work we prove the existence of a global attractor for a $p(x)$-Laplacian inclusion of the form $\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \alpha|u|^{p(x)-2}u \in F(u) + h$, $\alpha = 0, 1$.

RÉSUMÉ

Dans ce travail, nous prouvons l'existence d'un attracteur global d'une inclusion avec $p(x)$-Laplacien de la forme $\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \alpha|u|^{p(x)-2}u \in F(u) + h$, $\alpha = 0, 1$.

1. Introduction

Mathematical models with variable exponents appear in physical problems like electrorheological fluids (see [5,10,11]), image processing (see [1,4,6]), and porous medium equations (see [2,3,17]). We also refer the reader to [7] for an overview of differential equations with variable exponents. However, until now few works have appeared in the literature about global attractors for evolution problems involving variable exponents (see [9,12,15,13]).

Let us consider the following two problems:

$$(P1)\quad \begin{cases} \dfrac{\partial u}{\partial t}(t) - \operatorname{div}\big(\big|\nabla u(t)\big|^{p(x)-2}\nabla u(t)\big) \in F\big(u(t)\big) + h, \quad t > 0, \\ u(0) = u_0 \end{cases}$$

under homogeneous Dirichlet boundary conditions, and

$$(P2)\quad \begin{cases} \dfrac{\partial u}{\partial t}(t) - \operatorname{div}\big(\big|\nabla u(t)\big|^{p(x)-2}\nabla u(t)\big) + \big|u(t)\big|^{p(x)-2}u(t) \in F\big(u(t)\big) + h, \quad t > 0, \\ u(0) = u_0 \end{cases}$$

under homogeneous Neumann boundary conditions, where $p(\cdot) \in C(\bar{\Omega})$, $p^- := \inf p(x) > 2$, $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, is a bounded smooth domain, $h, u_0 \in H := L^2(\Omega)$, $F : \mathscr{D}(F) \subset L^2(\Omega) \to \mathscr{P}(L^2(\Omega))$, given by $F(y(\cdot)) = \{\xi(\cdot) \in L^2(\Omega)\colon \xi(x) \in f(y(x))$ x-a.e. in $\Omega\}$ with $f : \mathbb{R} \to \mathscr{C}_v(\mathbb{R})$ ($\mathscr{C}_v(\mathbb{R})$ is the set of all nonempty, bounded, closed, convex subsets of $\mathbb{R}$) be a multivalued map. Assume that f is Lipschitz, i.e., $\exists C \geqslant 0$ such that $\operatorname{dist}_{\mathscr{H}}(f(x), f(z)) \leqslant C\|x - z\|$, $\forall x, z \in \mathbb{R}$. Consequently, the map $F(u) + h$ has values in $\mathscr{C}_v(L^2(\Omega))$ and is Lipschitz.

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The paper is organized as follows. In Section 2 we present properties of the operators. In Section 3 we establish and prove our results on the existence of global attractors for the $p(x)$-Laplacian inclusions.

2. Properties of the operators

In [15,14] it is proved that the operator $Au := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the realization of the operator $A_1 : V \to V^*$, $V := W_0^{1,p(x)}(\Omega)$, $A_1u(v) := \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)\,dx$, i.e., $A(u) = A_1u$, if $u \in \mathscr{D}(A) := \{u \in V;\ A_1u \in H\}$ and is a maximal monotone operator in H. Besides, A is the subdifferential of a proper, convex and lower semi-continuous function $\varphi_A : H \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\varphi_A(u) := \begin{cases} \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx, & \text{if } u \in V,\\ +\infty, & \text{otherwise.}\end{cases}$$

Moreover, we have the following properties of the operator:

Lemma 2.1. *(See [14].)*

$$\langle Au, u\rangle_{V^*,V} \geqslant \begin{cases} \|u\|_V^{p^+}, & \text{if } \|u\|_V \leqslant 1,\\ \|u\|_V^{p^-}, & \text{if } \|u\|_V \geqslant 1,\end{cases} \quad \text{where } p^+ := \sup_{x\in\Omega} p(x).$$

In [16] it is proved that the operator $Bu := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u$ is the realization of the operator $B_1 : X \to X^*$ with $X := W^{1,p(x)}(\Omega)$, $B_1u(v) := \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)\,dx$, i.e., $B(u) = B_1u$, if $u \in \mathscr{D}(B) := \{u \in X;\ B_1u \in H\}$ and is a maximal monotone operator in H. Besides, B is the subdifferential of a proper, convex and lower semi-continuous function $\varphi_B : H \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\varphi_B(u) := \begin{cases} [\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx], & \text{if } u \in X,\\ +\infty, & \text{otherwise.}\end{cases}$$

Moreover, we have the following properties of the operator:

Lemma 2.2. *(See [16].)*

$$\langle Bu, u\rangle_{X^*,X} \geqslant \begin{cases} \frac{1}{2^{p^+-1}}\|u\|_X^{p^+}, & \text{if } \|u\|_X \leqslant 1,\\ \frac{1}{2^{p^--1}}\|u\|_X^{p^-}, & \text{if } \|u\|_{p(x)} \geqslant 1 \text{ and } \|\nabla u\|_{p(x)} \geqslant 1,\\ \|\nabla u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}, & \text{if } \|u\|_{p(x)} \leqslant 1 \text{ and } \|\nabla u\|_{p(x)} \geqslant 1,\\ \|\nabla u\|_{p(x)}^{p^+} + \|u\|_{p(x)}^{p^-}, & \text{if } \|u\|_{p(x)} \geqslant 1 \text{ and } \|\nabla u\|_{p(x)} \leqslant 1,\end{cases}$$

where $\|u\|_{p(x)} := \inf\{\lambda > 0;\ \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \leqslant 1\}$.

3. Existence of the global attractors

The following two propositions follow from Lemma 5 and Lemma 6 in [8].

Proposition 3.1. *The inclusion in $(P1)$ defines a strict multivalued semigroup (or strict m-semiflow) $G_1(t,\cdot) : H \to \mathscr{P}(H)$ where $G_1(t,u_0)$ is the set of all integral solutions of $(P1)$ beginning at $u_0 \in H$ valuated at time t.*

Proposition 3.2. *(See [16].) The inclusion in $(P2)$ defines a strict multivalued semigroup (or strict m-semiflow) $G_2(t,\cdot) : H \to \mathscr{P}(H)$ where $G_2(t,u_0)$ is the set of all integral solutions of $(P2)$ beginning at $u_0 \in H$ valuated at time t.*

Let us consider the following condition:

$(\mathscr{H})$ The sets $M_K := \{u \in D(\varphi):\ \|u\|_H \leqslant K,\ \varphi(u) \leqslant K\}$ are compact in H for any $K > 0$.

We intend to use the following:

Theorem 3.1. *(See [8].) Let $(\mathcal{H})$ be satisfied. Suppose that there exist $\delta > 0$, $M > 0$ such that $\forall u \in \mathcal{D}(\partial\varphi)$, $\|u\| \geqslant M$, $\forall y \in -\partial\varphi(u) + F(u) + h$,*

$$(y, u) \leqslant -\delta. \tag{1}$$

Then the multivalued semigroup G has a global attractor R. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.

Now, we establish our result:

Theorem 3.2. *The multivalued semigroup associated with problem $(P1)$ has a global attractor R_1. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.*

Proof. First, we will to prove that the condition $(\mathcal{H})$ is satisfied. Indeed, since $V \Subset H$ and

$$M_K := \left\{u \in \mathcal{D}(\varphi_A);\ \|u\|_H \leqslant K,\ \varphi_A(u) \leqslant K\right\} = \overline{M_K},$$

it is sufficient to show that for each $K > 0$, M_K is a bounded set in V. Let $K > 0$ and $u \in M_K$. Then, $\langle Au, u\rangle_{V^*,V} \leqslant Kp^+$. From Lemma 2.1, $\|u\|_V \leqslant \max\{[Kp^+]^{\frac{1}{p^-}}, [Kp^+]^{\frac{1}{p^+}}\}$. So, the condition $(\mathcal{H})$ is satisfied. Now, we intend to show that the condition (1) in Theorem 3.1 is satisfied. Let $u \in \mathcal{D}(A)$, $\xi \in F(u)$. Since the map f is Lipschitz and has values in $\mathcal{C}_v(\mathbb{R})$ it is easy to see that there exist $D_1, D_2 \geqslant 0$ such that $\sup_{y\in f(s)} |y| \leqslant D_1 + D_2|s|$, $\forall s \in \mathbb{R}$. Consequently, there are constants $k_1, k_2 > 0$ such that $\|\xi + h\|_H \leqslant k_1\|u\|_H + k_2$, $\forall \xi \in F(u)$. Using the immersion $V \subset H$, we have that $\|u\|_H \leqslant \sigma\|u\|_V$ for some $\sigma > 1$. Using Lemma 2.1, we obtain $\langle Au, u\rangle_{V^*,V} \geqslant (\frac{1}{\sigma})^{p^-}\|u\|_H^{p^-}$ for $\|u\|_H \geqslant \sigma$. Then, using the Cauchy–Schwarz and Young inequalities, we get $\langle -Au + \xi + h, u\rangle_{V^*,V} \leqslant -(\frac{1}{\sigma})^{p^-}\|u\|_H^{p^-} + k_1\|u\|_H^2 + k_2\|u\|_H \leqslant -\frac{1}{2\sigma^{p^-}}\|u\|_H^{p^-} + k_3$ for $\|u\|_H \geqslant \sigma$, with $k_3 := \frac{k_1^\alpha}{\alpha\epsilon_0^\alpha} + \frac{k_2^{q^-}}{q^-\epsilon_0^{q^-}}$, where $\frac{2}{p^-} + \frac{1}{\alpha} = 1$, $\frac{1}{p^-} + \frac{1}{q^-} = 1$ and $\epsilon_0 > 0$ is such that $\frac{2}{p^-}\epsilon_0^{p^-/2} + \frac{1}{p^-}\epsilon_0^{p^-} < \frac{1}{2\sigma^{p^-}}$. Considering $M := \max\{[2\sigma^{p^-}(1 + k_3)]^{1/p^-}, \sigma\} > 0$ and $\delta := 1$, we have $\langle -Au + \xi + h, u\rangle_{V^*,V} \leqslant -\delta$ for all $u \in \mathcal{D}(A)$ with $\|u\|_H > M$. So, condition (1) is satisfied and the result follows from Theorem 3.1. □

Theorem 3.3. *The multivalued semigroup associated with problem $(P2)$ has a global attractor R_2. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.*

Proof. Let $K > 0$ and $u \in M_K = \{u \in \mathcal{D}(\varphi_B);\ \|u\|_H \leqslant K,\ \varphi_B(u) \leqslant K\}$. As a consequence of Lemma 2.2, we get $\|u\|_X \leqslant \max\{[2p^+K2^{(p^- -1)}]^{\frac{1}{p^- -1}} + 1,\ [2^{p^+}p^+K]^{\frac{1}{p^+}}\}$. So, the condition $(\mathcal{H})$ is satisfied. The rest of the proof is completely analogous to the proof of Theorem 3.2, but here we use Lemma 2.2 to show that $\langle Bu, u\rangle_{X^*,X} \geqslant \min\{\frac{1}{\rho^{p^-}(2^{p^-}-1)}, \frac{1}{\gamma^{p^-}}\}\|u\|_H^{p^-}$ for $\|u\|_H \geqslant \gamma$, where $\gamma > 1$ is such that $\|u\|_H \leqslant \gamma\|u\|_{p(x)}$ and $\rho > 1$ is such that $\|u\|_H \leqslant \rho\|u\|_X$. □

Corollary 3.4. *The global attractors R_1, R_2 are bounded in V and X, respectively.*

Proof. Let $T > 0$. Since R_1 is negatively semi-invariant, we have $R_1 \subset G(t, R_1)$, $\forall t \geqslant 0$. In particular, $R_1 \subset G(T, R_1)$. From Corollary 3 in [8], there exists $K > 0$ such that $G(T, R_1) \subset M_K$. As M_K is bounded in V and $R_1 \subset M_K$, we obtain that R_1 is bounded in V. Analogously for R_2. □

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