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Hypergroupoids and C*-algebras

Hypergroupoïdes et C-algèbres*

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ABSTRACT

Let G be a locally compact groupoid. If X is a free and proper G-space, then $(X * X)/G$ is a groupoid equivalent to G. We consider the situation where X is proper, but no longer free. The formalism of groupoid C*-algebras and their representations is suitable to attach C*-algebras to this new object.

RÉSUMÉ

Soit G un groupoïde localement compact. Si X est un G-espace qui est libre et propre, alors $(X * X)/G$ est un groupoïde équivalent à G. On considère la situation où X est seulement propre. Le formalisme des C*-algèbres de groupoïdes permet d'associer des C*-algèbres à ce nouvel objet.

Version française abrégée

La C*-catégorie (au sens de [4]) d'un groupoïde localement compact G a été définie dans [7] à partir des G-espaces principaux mesurés (X, α), où X est un G-espace libre et propre et α est un système de mesures invariant pour l'application moment $r_X : X \to G^{(0)}$. On remarque dans cette note qu'on peut élargir cette C*-catégorie en considérant tous les G-espaces propres mesurés. Explicitement, supposons que G soit un groupoïde localement compact avec un système de Haar λ et que X soit un G-espace localement compact propre avec un système de mesures α sur les fibres de l'application moment $r_X : X \to G^{(0)}$, continu et invariant. On note $X * X$ l'espace des couples $(x, y) \in X \times X$ tels que $r_X(x) = r_X(y)$, $(X * X)/G$ le quotient par l'action diagonale de G et $[x, y]$ la classe de (x, y). Si le G-espace X est libre, $(X * X)/G$ est un groupoïde ; sinon, c'est un hypergroupoïde (dont on ne formalisera pas ici la définition). Dans tous les cas, $C_c((X * X)/G)$ est une algèbre involutive et $C_c(X)$ est un bimodule pré-hilbertien pour les opérations suivantes : avec $h \in C_c(G)$, $\xi, \eta \in C_c(X)$ et $f, g \in C_c((X * X)/G)$:

$$\begin{aligned} &\xi f(y) = \int \xi(x) f[x, y]\, \mathrm{d}\alpha(x) && \langle \xi, \eta\rangle[x, y] = \int \overline{\xi\left(\gamma^{-1}x\right)}\eta\left(\gamma^{-1}y\right) \mathrm{d}\lambda(\gamma) \\ &h\xi(x) = \int h(\gamma)\xi\left(\gamma^{-1}x\right) \mathrm{d}\lambda(\gamma) && \langle\langle \xi, \eta\rangle\rangle(\gamma) = \int \xi(x)\overline{\eta\left(\gamma^{-1}x\right)}\, \mathrm{d}\alpha(x) \\ &f^*[y, x] = \overline{f[x, y]} && f * g[x, z] = \int f[x, y] g[y, z]\, \mathrm{d}\alpha(y) \end{aligned} \tag{1}$$

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Théorème 0.1. *On suppose G à base dénombrable d'ouverts. Les représentations de l'algèbre involutive $C_c(G)$ qui sont non dégénérées et continues pour la topologie limite inductive se prolongent au triplet $(C_c(G), C_c(X), C_c((X * X)/G))$.*

Comme dans [7], la démonstration repose sur le théorème de désintégration des représentations. On obtient la norme pleine et la norme réduite de $C_c((X * X)/G)$ en prenant respectivement la représentation universelle et la représentation régulière de $C_c(G)$. On définit les C*-algèbres $C^*((X * X)/G)$ et $C_r^*((X * X)/G)$ de l'hypergroupoïde $(X * X)/G$ comme les C*-complétions relatives à ces normes.

Les paires (G, K), où K est un sous-groupe compact d'un groupe localement compact G fournissent des exemples classiques d'hypergroupes qui rentrent dans le cadre ci-dessus (avec $X = G/K$ et α la mesure invariante). On obtient les C*-algèbres $C^*(K\backslash G/K)$ et $C_r^*(K\backslash G/K)$. Si (G, K) est la complétion de Schlichting d'une paire de Hecke (Γ, Γ_0) comme dans [8], $C_r^*(K\backslash G/K)$ s'identifie à la C*-algèbre de cette paire de Hecke.

Considérons maintenant une paire (G, K) où K est un sous-groupoïde fermé d'un groupoïde localement compact. On suppose, de plus, que $H^{(0)} = G^{(0)}$, K est propre, l'application $r : G/K \to G^{(0)}$ admet un système de mesures invariant α et G possède un système de Haar λ. Alors $(X = G/K, \alpha)$ est un G-espace propre mesuré. Le théorème ci-dessus permet de définir les C*-algèbres pleine et réduite de l'hypergroupoïde $(X * X)/G = K\backslash G/K$. Dans la section 1 de [6], les auteurs, motivés par la construction de C*-algèbres de Hecke, considèrent le cas où $G = \Gamma \ltimes Y$ est le groupoïde de l'action d'un groupe Γ sur un espace Y et $H = \Lambda \ltimes Y$ où Λ est un sous-groupe de Γ qui agit proprement sur Y. L'article [5], qui propose une définition d'une paire de Gelfand dans le cadre des groupoïdes, considère aussi l'algèbre de convolution $C_c(K\backslash G/K)$ dans le cas où K est un sous-groupoïde compact d'un groupoïde localement compact G.

1. Introduction

This note stems from the elementary observation that the C*-category of a groupoid G defined in [7] can be extended from principal G-spaces to proper G-spaces. When X is a principal locally compact G-space with invariant r-system α, one can construct the $*$-algebra $(\alpha, \alpha)_c$ and its C*-completion (α, α); it is the C*-algebra of the locally compact groupoid $(X * X)/G$ equipped with the Haar system induced by α. When X is only proper, the same formulas define the $*$-algebra $(\alpha, \alpha)_c$ and its C*-completion (α, α); however, $(X * X)/G$ is no longer a groupoid, but a hypergroupoid. Objects like $(X * X)/G$ generalize both hypergroups (when the G-space X is transitive) and groupoids (when X is free). While convolution algebras of measures are commonly associated with hypergroups, our construction gives convolution algebras of functions and C*-algebras. It also covers the construction of C*-algebras from Hecke pairs as in [2,8]. In fact, the observation that $(X * X)/G$ is no longer a groupoid when X is not a free G-space but that its convolution algebra can still be defined appears in this context (see [6,3]). There, it is usual to introduce the reduced norm, while the existence of a maximal norm is problematic. Our framework provides natural maximal and reduced norms on the hypergroupoids we consider.

2. The C*-category of a groupoid

We review the framework and the main results of [7], but assuming that the G-spaces are proper and no longer free. For the sake of simplicity, we consider here an untwisted groupoid G. Given a topological groupoid G (with unit space $G^{(0)}$ and range and source maps r and s), a left G-space is a topological space X endowed with a continuous map $r_X : X \to G^{(0)}$, assumed to be open and onto, and a continuous action map $G * X \to X$, where $G * X$ is the subspace of composable pairs, i.e. $(\gamma, x) \in G \times X$ such that $s(\gamma) = r_X(x)$, sending (γ, x) to γx in such a way that $(\gamma\gamma')x = \gamma(\gamma' x)$ for all composable triples (γ, γ', x) and $ux = x$ if u is a unit. One says that the G-space X is proper [resp. free] if the map $G * X \to X \times X$ sending (γ, x) to $(\gamma x, x)$ is proper [resp. injective]. We shall assume that the groupoid G and the G-space X are locally compact (and Hausdorff for the sake of simplicity). If X is a proper locally compact G-space, then the quotient space X/G is locally compact. The image of $x \in X$ in the quotient space is denoted by $[x]$. If X, Y are G-spaces, we endow the space $X * Y$ of pairs (x, y) such that $r_X(x) = r_Y(y)$ with the diagonal action $\gamma(x, y) = (\gamma x, \gamma y)$. It is proper as soon as one of the factors is proper. Given a G-space X, an r_X-system of measures is a family $\alpha = (\alpha^u)_{u \in G^{(0)}}$ where α^u is a Radon measure on $X^u = r^{-1}(\{u\})$ with full support. We say that α is continuous if for all $f \in C_c(X)$, i.e. $f : X \to \mathbf{C}$ continuous with compact support, the function $u \mapsto \int f \, d\alpha^u$ is continuous on $G^{(0)}$. We say that α is invariant if for all $\gamma \in G$, $\gamma\alpha^{s(\gamma)} = \alpha^{r(\gamma)}$. The objects of our category are measured proper G-spaces (X, α), i.e. proper G-spaces X endowed with a continuous invariant r_X-system of measures α; we shall often omit the G-space X and write α instead of (X, α). Before defining the C*-category $C^*(G)$, we first define the $*$-category (in the sense of [4]) $C_c(G)$: given two measured proper G-spaces (X, α) and (Y, β), its set of arrows $(\alpha, \beta)_c$ consists of triples (α, f, β), where $f \in C_c((X * Y)/G)$. It is a complex vector space. Moreover, given measured proper G-spaces (X, α), (Y, β), (Z, γ) and $f \in C_c((X * Y)/G)$, $g \in C_c((Y * Z)/G)$, we define:

$$(\alpha, f, \beta)(\beta, g, \gamma) = (\alpha, f *_\beta g, \gamma)$$

where the convolution product is given by:

$$f *_\beta g[x, z] = \int f[x, y] g[y, z] \, d\beta^{r_X(x)}(y) \tag{2}$$

In this formula, a representative (x, z) has been fixed and $[x, z]$ denotes its class. The integration is over a compact set because the map $\varphi^x : Y^{r_X(x)} \to (X * Y)/G$ defined by $\varphi^x(y) = [x, y]$ is proper. The resulting integral depends on $[x, z]$ only because of the invariance of β. One also defines:

$$(\alpha, f, \beta)^* = \left(\beta, f^*, \alpha\right)$$

where the involution is given by $f^*[y, x] = \overline{f[x, y]}$.

Lemma 2.1. *(Cf. [7, Lemme 3.1].) These operations are well defined and turn $C_c(G)$ into a $*$-category.*

The next step is to define a C*-norm on the $*$-category $C_c(G)$. A unitary representation of G is a pair (m, H) where m is a transverse measure class [1, Definition A.1.19] and H is a Borel G-Hilbert bundle. We recall that m associates with (X, α) a measure class $m(\alpha)$ on X/G in a coherent fashion. A unitary representation of G defines by integration a representation of $C_c(G)$, that is, a functor into the W*-category of Hilbert spaces. It associates to the object (X, α) the Hilbert space $H(\alpha) = L^2(X/G, m(\alpha), X * H/G)$ and to the arrow (α, f, β) the operator $L(\alpha, f, \beta) : H(\beta) \to H(\alpha)$ defined by:

$$\left\langle \xi\sqrt{\mu}, L(\alpha, f, \beta)\eta\sqrt{\nu}\right\rangle = \int f[x, y]\left\langle\xi[x], \eta[y]\right\rangle\sqrt{(\mu \circ \dot{\beta}_1)(\nu \circ \dot{\alpha}_2)}[x, y]$$

where the sections $\xi\sqrt{\mu} \in H(\alpha)$ and $\eta\sqrt{\nu} \in H(\beta)$ are written as half-densities: μ [resp. ν] is a measure on X/G [resp. Y/G] in $m(\alpha)$ [resp. $m(\beta)$]. The systems of measures $\dot{\beta}_1$ and $\dot{\alpha}_2$ are induced by β and α respectively as in [7] or [1, Lemma A.1.3] for the proper case. For example, one has $\int f \, d\dot{\beta}_1^{[x]} = \int f[x, y] \, d\beta^{r_X(x)}(y)$. By definition, the measures $m_1 = \mu \circ \dot{\beta}_1$ and $m_2 = \nu \circ \dot{\alpha}_2$ are equivalent; their geometric mean is the measure $(dm_1/dm_2)^{1/2} \, dm_2$. Note that by Cauchy–Schwarz inequality,

$$\left\|L(\alpha, f, \beta)\right\| \le \max\left(\sup_x \int \left|f[x, y]\right| d\beta^{r_X(x)}(y), \sup_y \int \left|f[x, y]\right| d\alpha^{r_Y(y)}(x)\right)$$

The I-norm of f is defined as the right-hand side. Just as in [7], we have:

Theorem 2.2. *(Cf. [7, Proposition 3.5, Theorem 4.1].)*

(1) *Let (m, H) be a unitary representation of a locally compact groupoid G. Then the above formulas define a representation L of the $*$-category $C_c(G)$, called the integrated representation, which is continuous for the inductive limit topology and bounded for the I-norm.*
(2) *Let (G, λ) be a second countable locally compact groupoid with Haar system. Every representation of the $*$-algebra $C_c(G, \lambda)$ in a separable Hilbert space that is non-degenerate and continuous for the inductive limit topology is equivalent to an integrated representation.*

We deduce from this theorem that, given a locally compact groupoid with the Haar system (G, λ), the $*$-category $C_c(G)$ can be completed into a C*-category by defining the full C*-norm $\|(\alpha, f, \beta)\|$ as the supremum of $\|L(\alpha, f, \beta)\|$ over all unitary representations of G in separable Hilbert bundles. In particular, if (X, α) is a measured proper G-space, this defines the C*-algebra (α, α). If, moreover, X is a free G-space, $(X * X)/G$ is a groupoid equivalent to G; the algebra (α, α) is the full C*-algebra of this groupoid (endowed with the Haar system induced by α) and is Morita equivalent to $C^*(G, \lambda) = (\lambda, \lambda)$. If X is not free, $(X * X)/G$ is a hypergroupoid (the multiplication law is defined on its subsets rather than on its elements). It is still true that (λ, α) is a full C*-module over (α, α), but its algebra of compact operators is only an ideal of $C^*(G, \lambda)$. One has similar results with the regular representation and the reduced norm. If we identify $(G * X)/G = X$ through the map $(\gamma, x) \mapsto \gamma^{-1}x$, we obtain the various incarnations (1) of the formula (2).

3. Examples

1. Let K be a compact subgroup of a locally compact group G. The homogeneous space $X = G/K$ is a proper G-space equipped with an invariant measure α. Then, $(X \times X)/G$ is the double coset hypergroup $K\backslash G/K$. The full and the regular representations of G yield respectively the full and the reduced C*-algebras of this hypergroup.

2. Let Γ_0 be an almost normal subgroup of a discrete group Γ as in [2,8]. We equip Γ/Γ_0 with the counting measure. Since Γ_0 acts on Γ/Γ_0 with finite orbits, the convolution product is well defined on $C_c(\Gamma_0\backslash\Gamma/\Gamma_0)$, which becomes the Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$. Let (G, K) be the Schlichting completion of (Γ, Γ_0). Then $\mathcal{H}(\Gamma, \Gamma_0)$ can be identified with $C_c(K\backslash G/K)$ and we are in the situation of the first example.

3. A particular case of the next example, which generalizes the first example, is given in [6, Section 1]. Let (G, λ) be a locally compact groupoid with the Haar system and K a closed subgroupoid with $K^{(0)} = G^{(0)}$. Assume that K is a proper groupoid and that the map $r : G/K \to G^{(0)}$ has a G-invariant system of measures α. Then $(X = G/K, \alpha)$ is a measured proper G-space. Thus we can construct the hypergroupoid $(X * X)/G = K\backslash G/K$ and its full and its reduced C*-algebras. If K is principal, $(X * X)/G$ is a groupoid equivalent to G. The situation considered in [6] is the case of a semi-direct groupoid

$G = \Gamma \ltimes Y$ where a group Γ acts on a space Y and $H = \Lambda \ltimes Y$, where Λ is a subgroup of Γ acting properly on Y. The convolution algebra $C_c(K\backslash G/K)$ also appears in [5] (with K compact), where the authors give a groupoid version of a Gelfand pair.

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