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Differential geometry

On an isotropic property of anti-Kähler–Codazzi manifolds

Sur une propriété isotrope des variétés anti-Kähler–Codazzi

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ABSTRACT

We give a proof of the fact that an anti-Kähler–Codazzi manifold reduces to an isotropic anti-Kähler manifold if and only if the Ricci tensor field coincides with the Ricci* tensor field.

RÉSUMÉ

Nous donnons une preuve du fait qu'une variété de type anti-Kähler–Codazzi se réduit à une variété isotrope du même type si et seulement si le champ de tenseurs de Ricci coïncide avec le champ de tenseurs de Ricci*.

1. Introduction

In [3] we introduced the notion of an anti-Kähler–Codazzi manifold, for which a twin anti-Hermitian (Norden) metric satisfies the Codazzi equation. Such a structure gives rise to a new class of integrable anti-Hermitian structures, and we emphasize the importance of the Ricci and associated Ricci* tensor fields in the study of these manifolds. In this paper, we extend this study to other property of anti-Hermitian geometry, such as the isotropicity of anti-Hermitian structures.

We begin by collecting some basic materials that we need later. Let (M, J) be a $2n$-dimensional almost complex manifold, where J denotes its almost complex structure. We denote by $\Im^r_s(M)$ the module of all tensor fields of type (r, s) on M.

A semi-Riemannian metric g of neutral signature (n, n) is an anti-Hermitian (Norden) metric if:

$$g(JX, Y) = g(X, JY)$$

for any $X, Y \in \Im^1_0(M)$. An almost complex manifold (M, J) with an anti-Hermitian metric is referred to as an almost anti-Hermitian manifold. An anti-Kähler (Kähler–Norden) manifold can be defined as a triple (M, g, J), which consists of a smooth manifold M endowed with an almost complex structure J and an anti-Hermitian metric g such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. It is well known that the condition $\nabla J = 0$ is equivalent to the $\mathbb{C}$-holomorphicity (analyticity) of the anti-Hermitian metric g [2] (see p. 76), i.e. $\Phi_J g = 0$, where Φ_J is the Tachibana operator [4]: $(\Phi_J g)(X, Y, Z) = (L_{JX} g - L_X G)(Y, Z)$, and $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$ is the twin anti-Hermitian metric.

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2. Statement of the result

Let now (M, g, J) be an almost anti-Hermitian manifold. Then the pair (J, g) defines, as usual, the twin anti-Hermitian metric $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$. If the twin metric G satisfies the Codazzi equation:

$$(\nabla_X G)(Y, Z) - (\nabla_Y G)(X, Z) = 0 \tag{1}$$

or equivalently if the almost complex structure J satisfies:

$$(\nabla_X J)Y - (\nabla_Y J)X = 0$$

for any $X, Y \in \Im^1_0(M)$, then the triple (M, J, g) is called an anti-Kähler–Codazzi manifold (or *AKC*-space) [3]. Anti-Kähler–Codazzi manifolds are integrable almost anti-Hermitian manifolds (see [3]).

It is well known that the inner product in the vector space can be extended to an inner product in the tensor space. In fact, if T and L are tensors of type (r, s) with components $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ and $L^{k_1 \ldots k_r}_{l_1 \ldots l_s}$, then:

$$g(T, L) = g_{i_1 k_1} \ldots g_{i_r k_r} g^{j_1 l_1} \ldots g^{j_s l_s} T^{i_1 \ldots i_r}_{j_1 \ldots j_s} L^{k_1 \ldots k_r}_{l_1 \ldots l_s}.$$

If $T = L = \nabla J \in \Im^1_2(M)$, then the square norm $\|\nabla J\|^2$ of ∇J is defined by:

$$\|\nabla J\|^2 = g^{ij} g^{kl} g_{ms} (\nabla J)^m_{ik} (\nabla J)^s_{jl}.$$

An almost anti-Hermitian structure (M, g, J) is said to be isotropic anti-Kähler if $\|\nabla J\|^2 = 0$. The notion of isotropic Kähler structure is originally introduced in [1]. Some examples of isotropic anti-Kähler structures were given in [2]. From definition of isotropic anti-Kähler we have $\nabla J \neq 0$, in general. Conversely, from property $\nabla J = 0$, we immediately see that $\|\nabla J\|^2 = 0$, i.e. the anti-Kähler manifold is isotropic anti-Kähler.

Let now $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be the curvature operator of the Levi-Civita connection ∇ on an anti-Kähler–Codazzi manifold. Then the Ricci tensor S is defined as $S(X, Y) = \operatorname{trace}\{Z \to R(Z, X)Y\}$. The Ricci* tensor field $\overset{*}{S}$ of the anti-Kähler–Codazzi manifold is locally defined by:

$$\overset{*}{S}_{ij} = -R_{hjtl} G^{lh} J^t_i,$$

where $G^{lh} = J^l_s g^{sh} (G_{is} G^{sj} = -\delta^j_i)$ and R_{hjtl} are the covariant components of curvature tensor R. The fact that if an anti-Kähler–Codazzi manifold is anti-Kähler ($\nabla J = 0$), then $S = \overset{*}{S}$, is proved in [3]. The main result of this paper is the following.

Theorem 1. *An anti-Kähler–Codazzi manifold is isotropic anti-Kähler ($\|\nabla J\|^2 = 0$) if and only if $S = \overset{*}{S}$, where S and $\overset{*}{S}$ are the Ricci and Ricci* tensor fields, respectively.*

3. Proof of the theorem

Eq. (1) locally is equivalent to:

$$\nabla_k G_{ij} - \nabla_i G_{kj} = 0.$$

From here, using contraction with G^{ij}, we find:

$$(\nabla_i G_{kj}) G^{ij} = 0 \tag{2}$$

by virtue of $(\nabla_k G_{ij}) G^{ij} = 0$. In fact, since $G_{ij} G^{ij} = -\delta^i_i = -2n$, we have:

$$\begin{aligned}
&(\nabla_k G_{ij}) G^{ij} + G_{ij} \nabla_k G^{ij} = 0, \\
&(\nabla_k G_{ij}) G^{ij} + J^s_i g_{sj} \nabla_k \big(J^i_t g^{tj}\big) = (\nabla_k G_{ij}) G^{ij} + J^s_j g_{is} \nabla_k \big(J^i_t g^{tj}\big) \\
&\quad = (\nabla_k G_{ij}) G^{ij} + J^s_j g^{tj} \nabla_k \big(J^i_t g_{is}\big) = (\nabla_k G_{ij}) G^{ij} + G^{st} \nabla_k G_{st} \\
&\quad = 2(\nabla_k G_{ij}) G^{ij} = 0.
\end{aligned}$$

Applying ∇_l to Eq. (2), we obtain:

$$(\nabla_l \nabla_i G_{kj}) G^{ij} + (\nabla_k G_{ij}) \nabla_l G^{ij} = 0 \tag{3}$$

by virtue of (1). Using the contraction with g^{kl}, from (3) we have:

$$\begin{aligned} g^{kl}(\nabla_l \nabla_i G_{kj})G^{ij} + g^{kl}(\nabla_k G_{ij})\nabla_l G^{ij} &= (\nabla^k \nabla_k G_{ij})G^{ij} + g^{kl}(\nabla_k (g_{is} J_j^s))\nabla_l (g^{it} J_t^j) \\ &= (\nabla^k \nabla_k G_{ij})G^{ij} + g^{kl} g_{is}(\nabla_k J_j^s)(\nabla_l (g^{tj} J_t^i)) \\ &= (\nabla^k \nabla_k G_{ij})G^{ij} + g^{kl} g_{is} g^{tj}(\nabla_k J_j^s)(\nabla_l J_t^i) \\ &= (\nabla^k \nabla_k G_{ij})G^{ij} + \|\nabla J\|^2 = 0, \end{aligned} \tag{4}$$

where $\|\nabla J\|^2$ is the square norm of ∇J. In an anti-Kähler–Codazzi manifold, the complex structure J satisfies (see [3]):

$$\nabla_h \nabla_j J_i^h = (S_{jk} - \overset{*}{S}_{jk}) J_i^k,$$

which is equivalent to the following equation:

$$\nabla^k \nabla_k G_{ij} = (S_{jk} - \overset{*}{S}_{jk}) J_i^k \tag{5}$$

by virtue of:

$$\nabla_h \nabla_j J_i^h = \delta_h^s \nabla_s \nabla_j (G_{ik} g^{kh}) = g^{ks} \nabla_s \nabla_j G_{ik} = \nabla^k \nabla_k G_{ji}.$$

Substituting (5) into (4), we find:

$$(S_{jk} - \overset{*}{S}_{jk}) g^{jk} = \|\nabla J\|^2.$$

This means that a necessary and sufficient condition for an anti-Kähler–Codazzi manifold to reduce to an isotropic anti-Kähler manifold is that $S = \overset{*}{S}$. Thus the proof is complete.

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