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Number theory

Uniform lower bound for the least common multiple of a polynomial sequence ☆

Une borne inférieure uniforme pour le plus petit commun multiple d'une suite polynomiale

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ABSTRACT

Let n be a positive integer and $f(x)$ be a polynomial with nonnegative integer coefficients. We prove that $\operatorname{lcm}_{\lceil n/2\rceil \leqslant i \leqslant n}\{f(i)\} \geqslant 2^n$, except that $f(x)=x$ and $n=1,2,3,4,6$ and that $f(x)=x^s$, with $s \geqslant 2$ being an integer and $n=1$, where $\lceil n/2\rceil$ denotes the smallest integer, which is not less than $n/2$. This improves and extends the lower bounds obtained by M. Nair in 1982, B. Farhi in 2007 and S.M. Oon in 2013.

RÉSUMÉ

Soit n un entier $\geqslant 1$ et $f(x)$ un polynôme à coefficients entiers $\geqslant 0$. Nous démontrons que, à l'exception de certains cas explicites, on a $\operatorname{ppcm}_{\lceil n/2\rceil \leqslant i \leqslant n}\{f(i)\} \geqslant 2^n$, où $\lceil n/2\rceil$ dénote le plus petit entier $\geqslant n/2$. Ceci améliore, et étend, les bornes inférieures obtenues par M. Nair en 1982, B. Farhi en 2007 et S.M. Oon en 2013.

1. Introduction

The least common multiple of consecutive positive integers was first studied by Chebyshev, who made an important progress for the proof of the prime number theorem. Actually, Chebyshev [3] introduced the function $\psi(x) := \sum_{p^k \leqslant x} \log p = \log \operatorname{lcm}_{1\leqslant i\leqslant x}\{i\}$, where $x>0$ is a real number. From Chebyshev's work, one can derive that the prime number theorem is equivalent to the statement: $\psi(n) = \log \operatorname{lcm}(1,\dots,n) \sim n$ as n tends to infinity. Since then, the least common multiple of sequences of integers became popular. Bateman, Kalb and Stenger [2] gave an asymptotic formula of $\log \operatorname{lcm}_{1\leqslant i\leqslant n}\{b+ai\}$ as n tends to infinity, where $a \geqslant 1$ and $b \geqslant 0$ are coprime integers. Hong, Qian and Tan [11] got an asymptotic formula of the least common multiple of a sequence of products of linear polynomials. Qian and Hong [14] investigated the asymptotic behavior of the least common multiple of any consecutive arithmetic progression terms. Further, Farhi and Kane [6] and Hong and Qian [10] obtained some results on the least common multiple of consecutive arithmetic progression terms.

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Effective bounds for the least common multiple of integer sequences are given by several authors. Hanson [7] proved that $\mathrm{lcm}_{1\leqslant i\leqslant n}\{i\} < 3^n$ for any integer $n \geqslant 1$. Nair [12] showed that $\mathrm{lcm}_{1\leqslant i\leqslant n}\{i\} \geqslant 2^n$ for any integer $n \geqslant 7$. Lower bounds of the least common multiple of finite arithmetic progression have been investigated by Farhi [4,5], Hong and Feng [8], Hong and Kominers [9] and Wu et al. [15]. For the quadratic case, some results are also achieved. Farhi [5] provided a nontrivial lower bound for $\mathrm{lcm}_{1\leqslant i\leqslant n}\{i^2+1\}$. Oon [13] improved Farhi's lower bound by proving that $\mathrm{lcm}_{1\leqslant i\leqslant n}\{i^2+c\} \geqslant 2^n$ with c being a positive integer.

In this paper, we find surprisingly that 2^n is the uniform lower bound for the least common multiple of polynomial sequences of nonnegative integer coefficients. That is, we have the following result.

Theorem 1.1. *Let $n \geqslant 1$ be an integer and $f(x)$ be a polynomial of nonnegative integer coefficients. Then $\mathrm{lcm}_{\lceil n/2\rceil\leqslant i\leqslant n}\{f(i)\} \geqslant 2^n$ except that $f(x) = x$ and $n = 1, 2, 3, 4, 6$ and that $f(x) = x^s$ with $s \geqslant 2$ being an integer and $n = 1$, where $\lceil n/2\rceil$ stands for the smallest integer, which is not less than $n/2$.*

In particular, we have the following interesting result.

Corollary 1.2. *Let $n \geqslant 1$ be an integer and $f(x)$ be a polynomial of nonnegative integer coefficients. Then $\mathrm{lcm}_{1\leqslant i\leqslant n}\{f(i)\} \geqslant 2^n$ except that $f(x) = x$ and $n = 1, 2, 3, 4, 6$ and that $f(x) = x^s$ with $s \geqslant 2$ being an integer and $n = 1$.*

Evidently, if we take $f(x) = x$, then Corollary 1.2 becomes Nair's lower bound [12]. If one picks $f(x) = x^2 + c$, then Theorem 1.1 reduces to Oon's result [13].

The paper is organized as follows. In Section 2, we present some basic facts that are needed in the proof of our main result. Consequently, in Section 3, we prove three results about the least common multiple, and then show Theorem 1.1 as the conclusion of this paper.

2. Preliminaries

In this section, we show three lemmas that can be proved with a little effort and are needed in the proof of Theorem 1.1. Recall that a complex number is called an *algebraic integer* if it is a root of a monic polynomial of integer coefficients (see, for example, [1]).

Lemma 2.1. *Let $s \geqslant 1$ be an integer and $f(x) = \sum_{i=0}^{s} a_i x^i \in \mathbb{Z}[x]$ be a polynomial of degree s. If $\alpha_1, \ldots, \alpha_s$ are s roots of $f(x)$, then $a_s(\prod_{j\in\{1,\ldots,s\}\setminus\{i\}} \alpha_j)$ is an algebraic integer for each integer i with $1 \leqslant i \leqslant s$.*

Proof. Clearly, Lemma 2.1 is true if $s = 1$. We let $s \geqslant 2$ in what follows. Write $\beta_i := a_s(\prod_{j\in\{1,\ldots,s\}\setminus\{i\}} \alpha_j)$ for each integer i with $1 \leqslant i \leqslant s$. If at least two of $\alpha_1, \ldots, \alpha_s$ are zero, then $\beta_i = 0$ for each integer i with $1 \leqslant i \leqslant s$. So Lemma 2.1 holds in this case. If exactly one of $\alpha_1, \ldots, \alpha_s$ is zero, saying $\alpha_t = 0$ for some integer $t \in \{1, \ldots, s\}$, then $\beta_t = (-1)^{s-1}a_1 \in \mathbb{Z}$ and $\beta_i = 0$ for each integer i with $i \neq t$ and $1 \leqslant i \leqslant s$. Hence Lemma 2.1 is true in this case.

Assume now that none of $\alpha_1, \ldots, \alpha_s$ is zero. Fix an integer i with $1 \leqslant i \leqslant s$. Since $a_s \neq 0$, one has $\beta_i = a_s \frac{(-1)^s a_0/a_s}{\alpha_i} = (-1)^s \frac{a_0}{\alpha_i}$. Therefore, to show that β_i is an algebraic integer, it suffices to prove that $\frac{a_0}{\alpha_i}$ is an algebraic integer. From $f(\alpha_i) = 0$, one derives that:

$$\frac{a_0^{s-1}}{\alpha_i^s} f(\alpha_i) = \left(\frac{a_0}{\alpha_i}\right)^s + a_1\left(\frac{a_0}{\alpha_i}\right)^{s-1} + \cdots + a_{s-1}a_0^{s-2}\left(\frac{a_0}{\alpha_i}\right) + a_s a_0^{s-1} = 0.$$

This means that $\frac{a_0}{\alpha_i}$ is a root of the integer polynomial $g(x) = x^s + a_1 x^{s-1} + \cdots + a_{s-1}a_0^{s-2}x + a_s a_0^{s-1}$, from which it follows that $\frac{a_0}{\alpha_i}$ is an algebraic integer. Lemma 2.1 is proved in this case. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *For any positive integer $n \geqslant 7$, we have $\lceil n/2\rceil\binom{n}{\lceil n/2\rceil} > 2^n$.*

Proof. We prove Lemma 2.2 by induction on n. Evidently, $\lceil n/2\rceil\binom{n}{\lceil n/2\rceil} > 2^n$ holds for $n = 7$ and 8. Now let $n \geqslant 7$ and we assume that $\lceil n/2\rceil\binom{n}{\lceil n/2\rceil} > 2^n$ is true for the n case. Now we consider the $n+1$ case. One can easily check that:

$$\lceil (n+1)/2\rceil\binom{n+1}{\lceil (n+1)/2\rceil} = \begin{cases} 2\lceil n/2\rceil\binom{n}{\lceil n/2\rceil}, & \text{if } n \text{ is odd,} \\ (2\lceil n/2\rceil + 1)\binom{n}{\lceil n/2\rceil}, & \text{if } n \text{ is even.} \end{cases}$$

It then follows that $\lceil (n+1)/2\rceil\binom{n+1}{\lceil (n+1)/2\rceil} > 2^{n+1}$. Hence Lemma 2.2 holds for the $n+1$ case. Lemma 2.2 is proved. □

Lemma 2.3. *Let x be an indeterminate and let m and n be positive integers such that $m \leqslant n$. Then we have*:

$$\sum_{k=m}^{n}(-1)^{n-k}\binom{n-m}{k-m}\prod_{\substack{j=m\\ j\neq k}}^{n}(x-j)=(n-m)!. \tag{2.1}$$

Proof. We show (2.1) by induction on n. Obviously, (2.1) is true if $n=m$. Suppose that (2.1) holds for the $n-1$ case. Now we let $n>m$. We prove that (2.1) also holds for the n case. Since $(1-1)^{n-m}=0$, we have $\sum_{k=m}^{n-1}(-1)^{k-m}\binom{n-m}{k-m}=(-1)^{n-1-m}$. So by induction hypothesis, we get that:

$$\begin{aligned}(n-m)! &= (n-m)\sum_{k=m}^{n-1}(-1)^{n-1-k}\binom{n-1-m}{k-m}\prod_{\substack{j=m\\ j\neq k}}^{n-1}(x-j)\\ &= (n-m)\sum_{k=m}^{n-1}\frac{(-1)^{n-k}}{n-k}\binom{n-1-m}{k-m}\big((x-n)-(x-k)\big)\prod_{\substack{j=m\\ j\neq k}}^{n-1}(x-j)\\ &= \sum_{k=m}^{n-1}(-1)^{n-k}\binom{n-m}{k-m}\prod_{\substack{j=m\\ j\neq k}}^{n}(x-j)-\left(\prod_{j=m}^{n-1}(x-j)\right)\sum_{k=m}^{n-1}(-1)^{n-k}\binom{n-m}{k-m}\\ &= \sum_{k=m}^{n-1}(-1)^{n-k}\binom{n-m}{k-m}\prod_{\substack{j=m\\ j\neq k}}^{n}(x-j)+\binom{n-m}{n-m}\prod_{j=m}^{n-1}(x-j)\\ &= \sum_{k=m}^{n}(-1)^{n-k}\binom{n-m}{k-m}\prod_{\substack{j=m\\ j\neq k}}^{n}(x-j).\end{aligned}$$

Therefore (2.1) is true for the n case. This finishes the proof of Lemma 2.3. □

Remark. Lemma 2.3 can be proved using other methods. For example, one can prove it by using the fundamental theorem of algebra.

3. Proof of Theorem 1.1

In this section, we show Theorem 1.1. We begin with the following lemma.

Lemma 3.1. *Let $s \geqslant 1$ be an integer and $f(x)\in\mathbb{Z}[x]$ be a polynomial of degree s and with a_s as its leading coefficient. Then for any two positive integers m and n with $1\leqslant m\leqslant n$, we have*:

$$\operatorname{lcm}\big(f(m),f(m+1),\ldots,f(n)\big)\geqslant\frac{1}{(n-m)!}\prod_{k=m}^{n}\left|\frac{f(k)}{a_s}\right|^{\frac{1}{s}}.$$

Proof. If $f(k)=0$ for some integer k with $m\leqslant k\leqslant n$, then Lemma 3.1 is clearly true. In what follows, we assume that $f(k)\neq 0$ for all integers k with $m\leqslant k\leqslant n$.

Write $f(x)=\sum_{i=0}^{s}a_ix^i$. Suppose that $\alpha_1,\ldots,\alpha_s$ are s roots of $f(x)$. Then $f(x)=a_s(x-\alpha_1)\cdots(x-\alpha_s)$. It infers that $h_k(x):=(-1)^sf(k-x)=a_s\prod_{i=1}^{s}(x-(k-\alpha_i))\in\mathbb{Z}[x]$ is also a polynomial with the leading coefficient a_s and $k-\alpha_1,\ldots,k-\alpha_s$ are s roots of $h_k(x)$ for each integer k with $m\leqslant k\leqslant n$. So by Lemma 2.1, we know that $\frac{f(k)}{k-\alpha_i}=a_s\prod_{j\in\{1,\ldots,s\}\setminus\{i\}}(k-\alpha_j)$ is an algebraic integer for each pair (k,i) with $m\leqslant k\leqslant n$ and $1\leqslant i\leqslant s$. It follows that $\operatorname{lcm}(f(m),f(m+1),\ldots,f(n))/(k-\alpha_i)$ is an algebraic integer for each integer i with $1\leqslant i\leqslant s$. Since $f(k)\neq 0$ for all $m\leqslant k\leqslant n$, we have $k-\alpha_i\neq 0$ for all pairs (k,i) with $1\leqslant i\leqslant s$ and $m\leqslant k\leqslant n$. Then letting $x=\alpha_i(1\leqslant i\leqslant s)$ in (2.1), one deduces that:

$$\frac{(n-m)!}{\prod_{k=m}^{n}(k-\alpha_i)}=\sum_{k=m}^{n}(-1)^{k-m}\binom{n-m}{k-m}\frac{1}{k-\alpha_i}. \tag{3.1}$$

Multiplying both sides of (3.1) by $\operatorname{lcm}(f(m),\dots,f(n))$, we obtain that:

$$\mathcal{A}_i := (n-m)!\operatorname{lcm}\bigl(f(m),\dots,f(n)\bigr)\prod_{k=m}^{n}\frac{1}{k-\alpha_i}$$

is a nonzero algebraic integer, and so is the product $\mathcal{A} := \prod_{i=1}^{s}\mathcal{A}_i$. But one can easily derive that:

$$\mathcal{A} = \bigl((n-m)!\bigr)^s\bigl(\operatorname{lcm}\bigl(f(m),\dots,f(n)\bigr)\bigr)^s\prod_{k=m}^{n}\frac{a_s}{f(k)}, \tag{3.2}$$

which implies that $\mathcal{A}$ is a nonzero rational number. Thus $\mathcal{A}$ is a nonzero rational integer and so $|\mathcal{A}| \geqslant 1$. This, together with (3.2), concludes the desired result. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Let $f(x)$ be a polynomial of degree 2 and of nonnegative integer coefficients. Then for any integer $m \geqslant 2$, we have $\operatorname{lcm}(f(m-1), f(m)) \geqslant \frac{(m(m-1))^2}{2m-1}$.*

Proof. Since for any integer $m \geqslant 2$, we have $\gcd(f(m-1), f(m)) | (f(m) - f(m-1))$. This infers that $\gcd(f(m-1), f(m)) \leqslant f(m) - f(m-1)$. Write $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$, where $a_0, a_1 \geqslant 0$ and $a_2 \geqslant 1$. Then we have:

$$\begin{aligned}
\operatorname{lcm}\bigl(f(m-1), f(m)\bigr) &= \frac{f(m)f(m-1)}{\gcd(f(m-1), f(m))} \geqslant \frac{f(m)f(m-1)}{f(m)-f(m-1)}\\
&= \frac{(a_2m^2+a_1m+a_0)(a_2(m-1)^2+a_1(m-1)+a_0)}{(2m-1)a_2+a_1}\\
&\geqslant \frac{m^2(m-1)(a_2(m-1)+a_1)}{(2m-1)a_2+a_1}\\
&\geqslant \frac{m^2(m-1)(a_2(m-1)+a_1)}{(2m-1)a_2+\frac{2m-1}{m-1}a_1} = \frac{(m(m-1))^2}{2m-1}
\end{aligned}$$

as desired. This concludes the proof of Lemma 3.2. □

Lemma 3.3. *Let a and b be coprime positive integers. Then $\operatorname{lcm}(a, a+b, a+2b) = a(a+b)(a+2b)$ or $\frac{1}{2}a(a+b)(a+2b)$.*

Proof. Since a and b are coprime, we have $\gcd(a, a+b) | \gcd(a, (a+b)-a) = \gcd(a, b)$ and hence $\gcd(a, a+b) = 1$. Similarly, one has $\gcd(a+b, a+2b) = 1$ and $\gcd(a, a+2b) | \gcd(a, 2b)$. So $\gcd(a, a+2b) = 1$ or 2. Then the desired result follows immediately from the following well-known identity:

$$\operatorname{lcm}(a, a+b, a+2b) = \frac{a(a+b)(a+2b)\gcd(a, a+b, a+2b)}{\gcd(a, a+b)\gcd(a+b, a+2b)\gcd(a, a+2b)}.$$

So Lemma 3.3 is proved. □

We are now in a position to show Theorem 1.1.

Proof of Theorem 1.1. Since $f(x)$ is a polynomial with nonnegative integer coefficients, we may let $f(x) = a_sx^s + a_{s-1}x^{s-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$, where $a_i \geqslant 0$ and $a_s \geqslant 1$. Then for any integer $n \geqslant 7$, by Lemmas 3.1 and 2.2 and noting that $f(k) \geqslant a_sk^s$, we have:

$$\operatorname{lcm}_{\lceil n/2\rceil \leqslant i \leqslant n}\bigl\{f(i)\bigr\} \geqslant \frac{\prod_{k=\lceil n/2\rceil}^{n}\bigl|\frac{f(k)}{a_s}\bigr|^{\frac{1}{s}}}{(n-\lceil n/2\rceil)!} \geqslant \frac{\prod_{k=\lceil n/2\rceil}^{n}k}{(n-\lceil n/2\rceil)!} = \lceil n/2\rceil\binom{n}{\lceil n/2\rceil} > 2^n.$$

So it remains to check that Theorem 1.1 is true for all positive integers $n \leqslant 6$ in the following. First we consider the case $n = 1$. If $f(x)$ has at least two terms or $a_s \geqslant 2$, then $\operatorname{lcm}(f(\lceil n/2\rceil),\dots,f(n)) = f(1) = \sum_{i=0}^{s}a_i \geqslant 2$. Now let $2 \leqslant n \leqslant 6$. We divide the proof into the following three cases.

Case 1. $s \geqslant 3$. Then $\operatorname{lcm}(f(\lceil \frac{n}{2}\rceil),\dots,f(n)) \geqslant a_sn^s \geqslant n^s \geqslant n^3 > 2^n$ for each integer n with $2 \leqslant n \leqslant 6$. So Theorem 1.1 is true in this case.

Case 2. $s = 2$. Then $\operatorname{lcm}(f(\lceil \frac{n}{2}\rceil),\dots,f(n)) \geqslant f(n) \geqslant a_2n^2 \geqslant n^2 \geqslant 2^n$ for each integer n with $2 \leqslant n \leqslant 4$. On the other hand, by Lemma 3.2, we have $\operatorname{lcm}(f(\lceil \frac{n}{2}\rceil),\dots,f(n)) \geqslant \operatorname{lcm}(f(n-1), f(n)) \geqslant \frac{(n(n-1))^2}{2n-1} \geqslant 2^n$ for $n = 5, 6$. Theorem 1.1 is proved in this case.

Case 3. $s=1$. First let $a_0=0$, $a_1=1$ and $n=5$. Then $f(x)=x$. Hence $\text{lcm}_{\lceil \frac{n}{2}\rceil \leqslant i\leqslant n}\{f(i)\}=\text{lcm}_{\lceil \frac{5}{2}\rceil \leqslant i\leqslant 5}\{i\}=\text{lcm}(3,4,5)=60>2^5$ as required. Now let $a_0=0$ and $a_1\geqslant 2$. Then $f(x)=a_1x$. Denote $\mathcal{L}_n:=\text{lcm}_{\lceil \frac{n}{2}\rceil \leqslant i\leqslant n}\{i\}$. It is well known that $\mathcal{L}_n\geqslant 2^{n-1}$. Since $a_1\geqslant 2$ and $a_0=0$, one has $\text{lcm}_{\lceil \frac{n}{2}\rceil \leqslant i\leqslant n}\{f(i)\}=a_1\mathcal{L}_n\geqslant 2\mathcal{L}_n\geqslant 2^n$, as claimed.

Finally, let $a_0\geqslant 1$, $a_1\geqslant 1$ and $\gcd(a_0,a_1)=d$. One may write $a_1=da$ and $a_0=db$ for some coprime positive integers a and b. If $n=2$ and 3, then we have:

$$\begin{aligned}\text{lcm}_{\lceil \frac{n}{2}\rceil \leqslant i\leqslant n}\bigl\{f(i)\bigr\}&=\text{lcm}\bigl(f(n-1),f(n)\bigr)=\frac{f(n-1)f(n)}{\gcd(f(n-1),f(n))}\\&=\frac{f(n-1)f(n)}{\gcd(f(n)-f(n-1),f(n))}=\frac{(a_1(n-1)+a_0)(a_1n+a_0)}{\gcd(a_1,a_1n+a_0)}\\&=d\bigl(a(n-1)+b\bigr)(an+b)\geqslant n(n+1)>2^n\end{aligned}$$

as required. If $4\leqslant n\leqslant 6$, then by Lemma 3.3, one has:

$$\begin{aligned}\text{lcm}_{\lceil \frac{n}{2}\rceil \leqslant i\leqslant n}\bigl\{f(i)\bigr\}&\geqslant \text{lcm}\bigl(f(n-2),f(n-1),f(n)\bigr)=d\cdot \text{lcm}\bigl(a(n-2)+b,a(n-1)+b,an+b\bigr)\\&\geqslant \frac{1}{2}d\bigl(a(n-2)+b\bigr)\bigl(a(n-1)+b\bigr)(an+b)\geqslant \frac{1}{2}(n-1)n(n+1)>2^n,\end{aligned}$$

as desired.

This completes the proof of Theorem 1.1. □

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