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Partial differential equations

Lower bounds for the blow-up time in the higher-dimensional nonlinear divergence form parabolic equations

Bornes inférieures du temps d'explosion des solutions d'équations paraboliques non linéaires sous forme de divergence dans des espaces de grande dimension

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ABSTRACT

This paper deals with the blow-up of solutions to some nonlinear divergence form parabolic equations with nonlinear boundary conditions. We obtain a lower bound for the blow-up time of solutions in a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$.

RÉSUMÉ

L'article traite d'un problème d'explosion des solutions d'équations paraboliques non linéaires sous forme de divergence, avec des conditions aux limites non linéaires. On obtient une estimation d'une borne inférieure du temps d'explosion des solutions dans le cas d'un domaine borné $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$.

1. Introduction

In this paper, we obtain a lower bound for the blow-up time of solutions to the following problem:

$$\begin{cases} u_t = \sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} - f(u), & x \in \Omega,\ t > 0, \\ \sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}\nu_j = g(u), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$, is a convex bounded domain with smooth boundary, ν is the outward normal vector to $\partial\Omega$, $u_0(x)$ is the initial value and $(a^{ij}(x))_{n\times n}$ is a differentiable positive definite matrix. Moreover, we assume that the functions f and g are nonnegative and satisfy:

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$$f(\tau) \geqslant \gamma_1 \tau^p, \qquad g(\tau) \leqslant \gamma_2 \tau^q, \quad \tau \geqslant 0,$$

where $p > 1$, $q > 1$ and γ_1 and γ_2 are some positive constants.

Many papers in the literature are devoted to the bounds for the blow-up time in nonlinear parabolic problems, we refer the interested reader to [1,4,6–9] and the references therein.

In [5], the authors investigated the existence of global and blow-up solutions to problem (1). It was shown that if $p > q > 1$ and $2q < p + 1$, the nonnegative solutions to problem (1) does not blow-up in finite time. Moreover, they showed that under some reasonable conditions on f and g blow-up occurs. In addition, a lower bound for the blow-up time of solutions was obtained when $\Omega \subseteq \mathbb{R}^3$ is a bounded star shaped domain.

In this paper, we consider the case $2q \geqslant p + 1$, and obtain a lower bound for the blow-up time in a smooth bounded convex domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$.

In the next section, we will find a lower bound for the blow-up time, when the blow-up occurs.

2. A lower bound for the blow-up time

In this section we seek a lower bound for the blow-up time T in some appropriate measure. The idea of the proof of the following theorem came from [1].

Theorem 2.1. *Let $u(x,t)$ be the nonnegative classical solution to problem (1) in a smooth bounded convex domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$. Moreover, we assume that the nonnegative functions f and g satisfy*

$$f(\tau) \geqslant \gamma_1 \tau^p, \qquad g(\tau) \leqslant \gamma_2 \tau^q, \quad \tau \geqslant 0, \tag{2}$$

where $p > 1$, $q > 1$ and $2q \geqslant p + 1$ and γ_1 and γ_2 are some positive constants. Define

$$\Phi(t) = \int_\Omega u^{2k}\,\mathrm{d}x, \tag{3}$$

where $k > \max\{2(n-2)(q-1), \frac{q}{2} - 1, 1\}$ is a parameter. If $u(x,t)$ blows up at finite time T, then T is bounded from below by

$$\int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}\xi}{k_2 + k_1\xi + k_6\xi^{\frac{3(n-2)}{3n-8}}}, \tag{4}$$

where k_1, k_2 and k_6 are positive constants that will be determined later.

Proof. Since $(a^{ij}(x))_{n\times n}$ is a positive definite matrix, then there exists $\theta > 0$ such that for all $\eta \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n a^{ij}(x)\eta_i\eta_j \geqslant \theta|\eta|^2. \tag{5}$$

Now, we compute:

$$\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= 2k\int_\Omega u^{2k-1}u_t\,\mathrm{d}x \\
&= 2k\int_\Omega u^{2k-1}\left(\sum_{i,j=1}^n \left(a^{ij}(x)u_{x_i}\right)_{x_j} - f(u)\right)\mathrm{d}x \\
&= -2k(2k-1)\int_\Omega u^{2k-2}\left(\sum_{i,j=1}^n a^{ij}(x)u_{x_i}u_{x_j}\right)\mathrm{d}x + 2k\int_{\partial\Omega} u^{2k-1}g(u)\,\mathrm{d}s - 2k\int_\Omega u^{2k-1}f(u)\,\mathrm{d}x \\
&\leqslant -2k(2k-1)\theta\int_\Omega u^{2k-2}|\nabla u|^2\,\mathrm{d}x + 2k\int_{\partial\Omega} u^{2k-1}g(u)\,\mathrm{d}s - 2k\int_\Omega u^{2k-1}f(u)\,\mathrm{d}x \\
&\leqslant -\frac{2(2k-1)\theta}{k}\int_\Omega \left|\nabla u^k\right|^2\mathrm{d}x + 2k\gamma_2\int_{\partial\Omega} u^{2k+q-1}\,\mathrm{d}s - 2k\gamma_1\int_\Omega u^{2k+p-1}\,\mathrm{d}x,
\end{aligned} \tag{6}$$

where we have used (5) and (2) in the last inequalities. By integrating the following identity over Ω,

$$\nabla\cdot\left(xu^{2k+q-1}\right) = nu^{2k+q-1} + (2k+q-1)u^{2k+q-2}(x.\nabla u),$$

we obtain:

$$\int_{\partial\Omega} u^{2k+q-1}\,ds \leqslant \frac{n}{\rho_0}\int_{\Omega} u^{2k+q-1}\,dx + \frac{(2k+q-1)d}{\rho_0}\int_{\Omega} u^{2k+q-2}|\nabla u|\,dx, \tag{7}$$

where

$$\rho_0 = \min_{\partial\Omega}(x.\nu) > 0, \qquad d = \max_{\bar{\Omega}}|x|.$$

Note that ρ_0 is positive since Ω is assumed to be convex. Next, applying the Young inequality for (7) twice yields:

$$\int_{\partial\Omega} u^{2k+q-1}\,ds \leqslant \frac{1}{2}\int_{\Omega} u^{2k}\,dx + \frac{1}{2}\left(\left(\frac{n}{\rho_0}\right)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\right)\int_{\Omega} u^{2k+2q-2}\,dx + \frac{\epsilon_1}{2k^2}\int_{\Omega}\left|\nabla u^k\right|^2 dx, \tag{8}$$

where ϵ_1 is a positive constant to be determined later. Substituting (8) into (6), we get:

$$\begin{aligned}\frac{d\Phi}{dt} \leqslant{}& \left(-\frac{2(2k-1)\theta}{k} + \frac{\gamma_2\epsilon_1}{k}\right)\int_{\Omega}\left|\nabla u^k\right|^2 dx + k\gamma_2\int_{\Omega} u^{2k}\,dx\\ &+ k\gamma_2\left(\left(\frac{n}{\rho_0}\right)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\right)\int_{\Omega} u^{2k+2q-2}\,dx - 2k\gamma_1\int_{\Omega} u^{2k+p-1}\,dx.\end{aligned} \tag{9}$$

Now, by using the Hölder and Young inequalities, respectively, we get:

$$\int_{\Omega} u^{2k+2q-2}\,dx \leqslant |\Omega|^{m_1}\left(\int_{\Omega} u^{\frac{k(2n-3)}{(n-2)}}\,dx\right)^{m_2} \leqslant m_1|\Omega| + m_2\int_{\Omega} u^{\frac{k(2n-3)}{(n-2)}}\,dx, \tag{10}$$

where

$$m_1 = 1 - \frac{(n-2)(2k+2q-2)}{k(2n-3)}, \qquad m_2 = \frac{(n-2)(2k+2q-2)}{k(2n-3)}.$$

Combining (10) with (9) gives:

$$\begin{aligned}\frac{d\Phi}{dt} \leqslant{}& \left(-\frac{2(2k-1)\theta}{k} + \frac{\gamma_2\epsilon_1}{k}\right)\int_{\Omega}\left|\nabla u^k\right|^2 dx + k\gamma_2\int_{\Omega} u^{2k}\,dx\\ &+ k\gamma_2\left(\left(\frac{n}{\rho_0}\right)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\right)m_2\int_{\Omega} u^{\frac{k(2n-3)}{(n-2)}}\,dx\\ &+ k\gamma_2\left(\left(\frac{n}{\rho_0}\right)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\right)m_1|\Omega| - 2k\gamma_1\int_{\Omega} u^{2k+p-1}\,dx.\end{aligned} \tag{11}$$

We now make use of Schwarz's inequality to the third term on the right-hand side of (11) as follows:

$$\int_{\Omega} u^{\frac{k(2n-3)}{(n-2)}}\,dx \leqslant \left(\int_{\Omega} u^{2k}\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega} u^{\frac{2k(n-1)}{n-2}}\,dx\right)^{\frac{1}{2}} \leqslant \left(\int_{\Omega} u^{2k}\,dx\right)^{\frac{3}{4}}\left(\int_{\Omega}\left(u^k\right)^{\frac{2n}{n-2}}\,dx\right)^{\frac{1}{4}}. \tag{12}$$

By using the Sobolev inequality in [2, Corollary IX.14, p. 168] or in [3, Corollary 9.14, p. 284], for $n \geqslant 3$, we get:

$$\left\|u^k\right\|^{\frac{n}{2(n-2)}}_{L^{\frac{2n}{n-2}}(\Omega)} \leqslant (c_s)^{\frac{n}{2(n-2)}}\left\|u^k\right\|^{\frac{n}{2(n-2)}}_{W^{1,2}(\Omega)} \leqslant c\left(\left\|\nabla u^k\right\|^{\frac{n}{2(n-2)}}_{L^2(\Omega)} + \left\|u^k\right\|^{\frac{n}{2(n-2)}}_{L^2(\Omega)}\right), \tag{13}$$

where c_s is a constant depending on Ω and n and:

$$c = \begin{cases} 2^{\frac{1}{2}}(c_s)^{\frac{3}{2}}, & \text{for } n = 3,\\ (c_s)^{\frac{n}{2(n-2)}}, & \text{for } n > 3.\end{cases}$$

By inserting (13) in (12), we obtain:

$$\int_\Omega u^{\frac{k(2n-3)}{(n-2)}}\,dx \leqslant c\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{3}{4}}\big(\|\nabla u^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}} + \|u^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}}\big)$$
$$= c\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{3}{4}}\Big(\int_\Omega |\nabla u^k|^2\,dx\Big)^{\frac{n}{4(n-2)}} + c\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{2n-3}{2(n-2)}}. \tag{14}$$

Now, we can use the Young inequality to get:

$$\int_\Omega u^{\frac{k(2n-3)}{(n-2)}}\,dx \leqslant \frac{c^{\frac{4(n-2)}{3n-8}}(3n-8)}{4(n-2)\epsilon_2^{\frac{n}{3n-8}}}\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{3(n-2)}{3n-8}} + \frac{n\epsilon_2}{4(n-2)}\int_\Omega |\nabla u^k|^2\,dx + c\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{2n-3}{2(n-2)}}, \tag{15}$$

where ϵ_2 is a positive constant to be determined later. By using the Hölder inequality, we can have:

$$\int_\Omega u^{2k+p-1}\,dx \geqslant |\Omega|^{-\frac{p-1}{2k}}\Big(\int_\Omega u^{2k}\,dx\Big)^{1+\frac{p-1}{2k}}. \tag{16}$$

Next, we can apply the Young inequality to the third term on the right-hand side of (15) to conclude:

$$\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{2n-3}{2(n-2)}} \leqslant \frac{(\epsilon_3)^{-\frac{m_3}{m_4}}}{m_3}\Big(\int_\Omega u^{2k}\,dx\Big)^{\frac{3(n-2)}{3n-8}} + \frac{\epsilon_3}{m_4}\Big(\int_\Omega u^{2k}\,dx\Big)^{1+\frac{p-1}{2k}}, \tag{17}$$

where

$$m_3 = \frac{2(n-2)(6k(n-2)-(2k+p-1)(3n-8))}{(3n-8)(2k(2n-3)-2(n-2)(2k+p-1))},$$
$$m_4 = \frac{2(n-2)(6k(n-2)-(3n-8)(2k+p-1))}{2k(6(n-2)^2-(2n-3)(3n-8))},$$

and ϵ_3 is a positive constant to be determined later. Combining (15), (16) and (17) with (11), we get:

$$\frac{d\Phi}{dt} \leqslant k_1\Phi - 2k\gamma_1|\Omega|^{-\frac{p-1}{2k}}\Phi^{1+\frac{p-1}{2k}} + k_2 + k_3\Phi^{\frac{3(n-2)}{3n-8}} + k_4\Phi^{\frac{2n-3}{2(n-2)}} + k_5\int_\Omega |\nabla u^k|^2\,dx$$
$$\leqslant k_1\Phi + \Big(-2k\gamma_1|\Omega|^{-\frac{p-1}{2k}} + \frac{k_4\epsilon_3}{m_4}\Big)\Phi^{1+\frac{p-1}{2k}} + k_2 + \Big(k_3 + \frac{k_4}{m_3}(\epsilon_3)^{-\frac{m_3}{m_4}}\Big)\Phi^{\frac{3(n-2)}{3n-8}} + k_5\int_\Omega |\nabla u^k|^2\,dx,$$

where

$$k_1 = k\gamma_2,$$
$$k_2 = k\gamma_2\Big[\Big(\frac{n}{\rho_0}\Big)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\Big]m_1|\Omega|,$$
$$k_3 = k\gamma_2 m_2\Big[\Big(\frac{n}{\rho_0}\Big)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\Big]\frac{(3n-8)}{4(n-2)}\frac{c^{\frac{4(n-2)}{3n-8}}}{\epsilon_2^{\frac{n}{3n-8}}},$$
$$k_4 = k\gamma_2\Big[\Big(\frac{n}{\rho_0}\Big)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\Big]m_2 c,$$
$$k_5 = \Big[k\gamma_2\Big(\Big(\frac{n}{\rho_0}\Big)^2 + \frac{(2k+q-1)^2d^2}{\rho_0^2\epsilon_1}\Big)m_2\Big]\frac{n\epsilon_2}{4(n-2)} + \frac{\gamma_2\epsilon_1}{k} - \frac{2(2k-1)\theta}{k}.$$

By choosing $\epsilon_1 > 0$ small enough, we can choose $\epsilon_2 > 0$ such that $k_5 = 0$. We also choose $\epsilon_3 > 0$ such that:

$$\epsilon_3 = \frac{2k\gamma_1 m_4}{k_4}|\Omega|^{-\frac{p-1}{2k}}.$$

Hence, we can write:

$$\frac{d\Phi}{dt} \leqslant k_2 + k_1\Phi + k_6\Phi^{\frac{3(n-2)}{3n-8}}, \tag{18}$$

where

$$k_6 = k_3 + \frac{k_4}{m_3}(\epsilon_4)^{-\frac{m_3}{m_4}}.$$

Then

$$\frac{\mathrm{d}\Phi}{k_2 + k_1\Phi + k_6\Phi^{\frac{3(n-2)}{3n-8}}} \leqslant 1. \tag{19}$$

Integrating of (19) from 0 to t, we obtain:

$$\int_{\Phi(0)}^{\Phi(t)} \frac{\mathrm{d}\xi}{k_2 + k_1\xi + k_6\xi^{\frac{3(n-2)}{3n-8}}} \leqslant t.$$

Passing to the limit as $t \to T^-$, we get:

$$\int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}\xi}{k_2 + k_1\xi + k_6\xi^{\frac{3(n-2)}{3n-8}}} \leqslant T.$$

Thus, the proof is complete. □

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