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Complex Analysis

Squares of positive (p, p)-forms

Carrés de (p, p)-formes positives

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ABSTRACT

We show that if α is a positive $(2, 2)$-form, then so is α^2. We also prove that this is no longer true for forms of higher degree.

RÉSUMÉ

Nous montrons que si α est une $(2, 2)$-forme positive alors α^2 l'est aussi. Nous prouvons également que ceci n'est plus vrai pour les formes de degré supérieur.

1. Introduction

Recall that a (p, p)-form α in $\mathbb{C}^n$ is called *positive* (we write $\alpha \geqslant 0$) if for $(1, 0)$-forms $\gamma_1, \dots, \gamma_{n-p}$ one has:

$$\alpha \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \dots \wedge i\gamma_{n-p} \wedge \bar{\gamma}_{n-p} \geqslant 0.$$

This is a natural geometric condition, positive (p, p)-forms are for example characterized by the following property: for every p-dimensional subspace V and a test function $\varphi \geqslant 0$, one has:

$$\int_V \varphi\alpha \geqslant 0.$$

It is well known that positive forms are real (that is $\bar{\alpha} = \alpha$) and if β is a $(1, 1)$-form then

$$\alpha \geqslant 0, \qquad \beta \geqslant 0 \quad \Rightarrow \quad \alpha \wedge \beta \geqslant 0. \tag{1}$$

It was shown by Harvey and Knapp [5] (and independently by Bedford and Taylor [1]) that (1) does not hold for all (p, p) and (q, q)-forms α and β, respectively. We refer to Demailly's book [2], pp. 129–132, for a nice and simple introduction to positive forms.

Dinew [3] gave an explicit example of $(2, 2)$-forms α, β in $\mathbb{C}^4$ such that $\alpha \geqslant 0$, $\beta \geqslant 0$ but $\alpha \wedge \beta < 0$. We will recall it in the next section. The aim of this note is to show the following, somewhat surprising result:

Theorem 1. *Assume that α is a positive $(2, 2)$-form. Then α^2 is also positive.*

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It turns out that this phenomenon holds only for $(2,2)$-forms:

Theorem 2. *For every $p \geqslant 3$, there exists a (p,p)-form α in $\mathbb{C}^{2p}$ such that $\alpha \geqslant 0$ but $\alpha^2 < 0$.*

We do not know if similar results hold for higher powers of positive forms.

The paper is organized as follows: in Section 2 we present Dinew's criterion for positivity of $(2,2)$-forms in $\mathbb{C}^4$, which reduces the problem to a certain property of 6×6 matrices. Further simplification reduces the problem to 4×4 matrices. We then solve it in Section 3. This is the most technical part of the paper. Higher degree forms are analyzed in Section 4, where a counterpart of Dinew's criterion is showed and Theorem 2 is proved.

2. Dinew's criterion

Without loss of generality we may assume that $n = 4$. Let $\omega_1, \dots, \omega_4$ be a basis of $(\mathbb{C}^4)^*$ such that:

$$dV := i\omega_1 \wedge \bar{\omega}_1 \wedge \cdots \wedge i\omega_4 \wedge \bar{\omega}_4 = \omega_1 \wedge \cdots \wedge \omega_4 \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_4 > 0.$$

Set

$$\begin{aligned} \Omega_1 &:= \omega_1 \wedge \omega_2, \qquad \Omega_2 := \omega_1 \wedge \omega_3, \qquad \Omega_3 := \omega_1 \wedge \omega_4, \\ \Omega_4 &:= \omega_2 \wedge \omega_3, \qquad \Omega_5 := -\omega_2 \wedge \omega_4, \qquad \Omega_6 := \omega_3 \wedge \omega_4. \end{aligned}$$

Then

$$\Omega_j \wedge \Omega_k = \begin{cases} \omega_1 \wedge \cdots \wedge \omega_4, & \text{if } k = 7-j, \\ 0, & \text{otherwise.} \end{cases}$$

With every $(2,2)$-form α we can associate a 6×6-matrix $A = (a_{jk})$ by

$$\alpha = \sum_{j,k} a_{jk} \Omega_j \wedge \bar{\Omega}_k.$$

For

$$\beta = \sum_{j,k} b_{jk} \Omega_j \wedge \bar{\Omega}_k$$

we have:

$$\alpha \wedge \beta = \sum_{j,k} a_{jk} b_{7-j,7-k}\, dV. \tag{2}$$

The key will be the following criterion from [3]:

Theorem 3. *$\alpha \geqslant 0$ if $\bar{z} A z^T \geqslant 0$ for all $z \in \mathbb{C}^6$ with $z_1 z_6 + z_2 z_5 + z_3 z_4 = 0$.*

Sketch of proof. For $\gamma_1 = b_1\omega_1 + \cdots + b_4\omega_4$, $\gamma_2 = c_1\omega_1 + \cdots + c_4\omega_4$, we have

$$\begin{aligned} i\gamma_1 \wedge \bar{\gamma}_1 \wedge i\gamma_2 \wedge \bar{\gamma}_2 &= \sum_{j,k,l,m=1}^{4} b_j \bar{b}_k c_l \bar{c}_m\, \omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m \\ &= \sum_{\substack{j<l \\ k<m}} (b_j c_l - b_l c_j)\overline{(b_k c_m - b_m c_k)} \omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m. \end{aligned}$$

It is now enough to show that the image of the mapping:

$$\begin{aligned} &\mathbb{C}^8 \ni (b_1, \dots, b_4, c_1, \dots, c_4) \\ &\quad \longmapsto (b_1c_2 - b_2c_1, b_1c_3 - b_3c_1, b_1c_4 - b_4c_1, b_2c_3 - b_3c_2, -b_2c_4 + b_4c_2, b_3c_4 - b_4c_3) \in \mathbb{C}^6 \end{aligned}$$

is precisely $\{z \in \mathbb{C}^6\colon\ z_1z_6 + z_2z_5 + z_3z_4 = 0\}$. Indeed, it is a well-known fact that the image of the Plücker embedding of the 4-dimensional Grassmannian $G(2,4)$ in $P(\Lambda^2\mathbb{C}^4) \simeq \mathbb{P}^5$ is the quadric defined by the above equation. □

Using Theorem 3 and an idea from [3], we can show:

Proposition 4. *The form*

$$\alpha_a = \sum_{j=1}^{6} \Omega_j \wedge \bar{\Omega}_j + a\,\Omega_1 \wedge \bar{\Omega}_6 + \bar{a}\,\Omega_6 \wedge \bar{\Omega}_1$$

is positive if and only if $|a| \leqslant 2$.

Proof. We have:

$$\bar{z}Az^T = |z|^2 + 2\,\mathrm{Re}(a\bar{z}_1 z_6) \geqslant 2|z_1 z_6| + 2|z_2 z_5 + z_3 z_4| + 2\,\mathrm{Re}(a\bar{z}_1 z_6).$$

If $z_1 z_6 + z_2 z_5 + z_3 z_4 = 0$ and $|a| \leqslant 2$ we clearly get $\bar{z}Az^T \geqslant 0$. If we take z_1, z_6 with $\bar{z}_1 z_6 = -\bar{a}$, $|z_1| = |z_6|$ and $z_2, \ldots, z_5$ with $z_2 z_5 + z_3 z_4 = -z_1 z_6$ then $\bar{z}Az^T = 2|a|(2 - |a|)$. □

By (2):

$$\alpha_a \wedge \alpha_b = 2\big(3 + \mathrm{Re}(a\bar{b})\big)\,dV.$$

Therefore, α_2, α_{-2} are positive, but $\alpha_2 \wedge \alpha_{-2} < 0$.

In view of Theorem 3, we see that Theorem 1 is equivalent to the following:

Theorem 5. *Let* $A = (a_{jk}) \in \mathbb{C}^{6\times 6}$ *be hermitian and such that* $\bar{z}Az^T \geqslant 0$ *for* $z \in \mathbb{C}^6$ *with* $z_1 z_6 + z_2 z_5 + z_3 z_4 = 0$. *Then*

$$\sum_{j,k=1}^{6} a_{jk} a_{7-j,7-k} \geqslant 0.$$

We will need the following technical reduction:

Lemma 6. *For every* $(2,2)$*-form* α *in* $\mathbb{C}^4$, *we can find a basis* $\omega_1, \ldots, \omega_4$ *of* $(\mathbb{C}^4)^*$ *such that*:

$$\alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_j = \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_2 \wedge \bar{\omega}_j = 0 \tag{3}$$

for $j = 3, 4$.

Proof. We may assume that $\alpha \neq 0$, then we can find $\omega_1, \omega_2 \in (\mathbb{C}^4)^*$ such that

$$\alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = \alpha \wedge i\omega_1 \wedge \bar{\omega}_1 \wedge i\omega_2 \wedge \bar{\omega}_2 \neq 0. \tag{4}$$

By V_1 denote the subspace spanned by ω_1, ω_2 and by V_2 the subspace of all $\omega \in (\mathbb{C}^4)^*$ satisfying (3) with ω_j replaced by ω. Then $\dim V_1 = 2$, $\dim V_2 \geqslant 2$, and by (4) we infer $V_1 \cap V_2 = \{0\}$, hence $(\mathbb{C}^4)^* = V_1 \oplus V_2$. □

3. Proof of Theorem 5

By Lemma 6 we may assume that the matrix from Theorem 5 satisfies

$$a_{26} = a_{36} = a_{46} = a_{56} = 0$$

and

$$a_{62} = a_{63} = a_{64} = a_{65} = 0.$$

Then

$$\sum_{j,k=1}^{6} a_{jk} a_{7-j,7-k} = \sum_{j,k=2}^{5} a_{jk} a_{7-j,7-k} + 2\big(a_{11}a_{66} + |a_{16}|^2\big).$$

Therefore Theorem 5 is in fact equivalent to the following result:

Theorem 7. *Let* $A = (a_{jk}) \in \mathbb{C}^{4\times 4}$ *be hermitian and such that* $\bar{z}Az^T \geqslant 0$ *for* $z \in \mathbb{C}^4$ *with* $z_1 z_4 + z_2 z_3 = 0$. *Then*

$$\sum_{j,k=1}^{4} a_{jk}\, a_{5-j,5-k} \geqslant 0. \tag{5}$$

Proof. Write

$$A = \begin{pmatrix} \lambda_1 & a & b & \alpha \\ \bar{a} & \lambda_2 & \beta & -c \\ \bar{b} & \bar{\beta} & \lambda_3 & -d \\ \bar{\alpha} & -\bar{c} & -\bar{d} & \lambda_4 \end{pmatrix}.$$

It satisfies the assumption of the theorem if and only if for every $z \in \mathbb{C}^4$ of the form $z = (1, \zeta, w, -\zeta w)$ one has $\bar{z}Az^T \geqslant 0$. We can then compute

$$\begin{aligned}\bar{z}Az^T &= \lambda_1 + 2\operatorname{Re}(a\zeta) + \lambda_2|\zeta|^2 \\ &\quad + 2\operatorname{Re}\big[\big(b - \alpha\zeta + \beta\bar{\zeta} + c|\zeta|^2\big)w\big] \\ &\quad + \big(\lambda_3 + 2\operatorname{Re}(d\zeta) + \lambda_4|\zeta|^2\big)|w|^2.\end{aligned}$$

Therefore A satisfies the assumption if $\lambda_j \geqslant 0$,

$$|a| \leqslant \sqrt{\lambda_1\lambda_2}, \qquad |b| \leqslant \sqrt{\lambda_1\lambda_3}, \qquad |c| \leqslant \sqrt{\lambda_2\lambda_4}, \qquad |d| \leqslant \sqrt{\lambda_3\lambda_4}, \tag{6}$$

and for every $\zeta \in \mathbb{C}$

$$\big|b - \alpha\zeta + \beta\bar{\zeta} + c|\zeta|^2\big|^2 \leqslant \big(\lambda_1 + 2\operatorname{Re}(a\zeta) + \lambda_2|\zeta|^2\big)\big(\lambda_3 + 2\operatorname{Re}(d\zeta) + \lambda_4|\zeta|^2\big). \tag{7}$$

In our case (5) is equivalent to

$$4\operatorname{Re}(a\bar{d} + b\bar{c}) \leqslant 2(\lambda_1\lambda_4 + \lambda_2\lambda_3) + 2\big(|\alpha|^2 + |\beta|^2\big).$$

We will in fact prove something more:

$$4\operatorname{Re}(a\bar{d} + b\bar{c}) \leqslant (\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3})^2 + \big(|\alpha| + |\beta|\big)^2. \tag{8}$$

Without loss of generality, we may assume that:

$$\operatorname{Re}(a\bar{d}) > 0, \qquad \operatorname{Re}(b\bar{c}) > 0,$$

for if for example $\operatorname{Re}(a\bar{d}) \leqslant 0$ then by (6)

$$4\operatorname{Re}(a\bar{d} + b\bar{c}) \leqslant 4\operatorname{Re}(b\bar{c}) \leqslant 4\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4} \leqslant (\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3})^2.$$

Set $u := \operatorname{Re}(a\bar{d})$ and $\zeta := -r\bar{d}/|d|$, where $r > 0$ will be determined later. Then we can write the right-hand side of (7) as follows:

$$\begin{aligned}&\left(\lambda_1 - \frac{2ur}{|d|} + \lambda_2 r^2\right)\big(\lambda_3 - 2r|d| + \lambda_4 r^2\big) \\ &\quad = \big(\lambda_1 + \lambda_2 r^2\big)\big(\lambda_3 + \lambda_4 r^2\big) + 4ur^2 - 2r\left[\lambda_1|d| + \lambda_3\frac{u}{|d|} + r^2\left(\lambda_2|d| + \lambda_4\frac{u}{|d|}\right)\right] \\ &\quad \leqslant \big(\lambda_1 + \lambda_2 r^2\big)\big(\lambda_3 + \lambda_4 r^2\big) + 4ur^2 - 4r^2(\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3})\sqrt{u} \\ &\quad = (\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3} - 2\sqrt{u})^2 r^2 + \big(\sqrt{\lambda_1\lambda_3} - \sqrt{\lambda_2\lambda_4}\, r^2\big)^2.\end{aligned}$$

For $r = (\frac{\lambda_1\lambda_3}{\lambda_2\lambda_4})^{1/4}$ from (7) we thus obtain:

$$\left|\frac{b}{r} + \frac{\bar{d}}{|d|}\alpha - \frac{d}{|d|}\beta + cr\right| \leqslant \sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3} - 2\sqrt{u}.$$

We also have:

$$\left|\frac{b}{r} + \frac{\bar{d}}{|d|}(\alpha - \bar{\beta}) + cr\right| \geqslant \left|\frac{b}{r} + cr\right| - |\alpha| - |\beta| \geqslant 2\sqrt{\operatorname{Re}(b\bar{c})} - |\alpha| - |\beta|$$

and therefore:

$$2\sqrt{\operatorname{Re}(a\bar{d})} + 2\sqrt{\operatorname{Re}(b\bar{c})} \leqslant \sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3} + |\alpha| + |\beta|.$$

To get (8), we can now use the following fact: if $0 \leqslant a_1 \leqslant x$, $0 \leqslant a_2 \leqslant x$ and $a_1 + a_2 \leqslant x + y$ then $a_1^2 + a_2^2 \leqslant x^2 + y^2$. This can be easily verified: if $a_1 + a_2 \leqslant x$ then $a_1^2 + a_2^2 \leqslant x^2$ and if $a_1 + a_2 \geqslant x$ then

$$x^2 + y^2 \geqslant x^2 + (a_1 + a_2 - x)^2 = a_1^2 + a_2^2 + 2x(x - a_1)(x - a_2). \qquad \square$$

4. (p, p)-Forms in $\mathbb{C}^{2p}$

In $\mathbb{C}^{2p}$ we choose the positive volume form:

$$dV := i\,dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge i\,dz_{2p} \wedge d\bar{z}_{2p} = dz_1 \wedge \cdots \wedge dz_{2p} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{2p}.$$

By $\mathcal{I}$ we will denote the set of subscripts $J = (j_1, \ldots, j_p)$ such that $1 \leqslant j_1 < \cdots < j_p \leqslant 2p$. For every $J \in \mathcal{I}$ there exists unique $J' \in \mathcal{I}$ such that $J \cup J' = \{1, \ldots, 2p\}$. We also denote $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and $\varepsilon_J = \pm 1$ is defined in such a way that:

$$dz_J \wedge dz_{J'} = \varepsilon_J\, dz_1 \wedge \cdots \wedge dz_{2p}.$$

Note that:

$$\varepsilon_J \varepsilon_{J'} = (-1)^p. \tag{9}$$

With every (p, p)-form α in $\mathbb{C}^{2p}$ we can associate an $N \times N$-matrix (a_{JK}), where

$$N = \sharp\mathcal{I} = \frac{(2p)!}{(p!)^2},$$

by

$$\alpha = \sum_{J,K} a_{JK}\, i\,dz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge i\,dz_{j_p} \wedge d\bar{z}_{k_p} = i^{p^2} \sum_{J,K} a_{JK}\, dz_J \wedge d\bar{z}_K \tag{10}$$

(note that $(-1)^{p(p-1)/2} i^p = i^{p^2}$). Then

$$\alpha^2 = \sum_{J,K} \varepsilon_J \varepsilon_K a_{JK} a_{J'K'}\, dV \tag{11}$$

and for $\gamma_1, \ldots, \gamma_p \in (\mathbb{C}^{2p})^*$

$$\alpha \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i\gamma_p \wedge \bar{\gamma}_p = \sum_{J,K} a_{JK}\, \gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J \wedge \overline{(\gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_K)}.$$

Therefore (a_{JK}) has to be positive semi-definite on the image of the Plücker embedding

$$\big((\mathbb{C}^{2p})^*\big)^p \ni (\gamma_1, \ldots, \gamma_p) \longmapsto \left(\frac{\gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J}{dz_1 \wedge \cdots \wedge dz_{2p}}\right)_{J \in \mathcal{I}} \in \mathbb{C}^N \tag{12}$$

which is well known to be a variety in $\mathbb{C}^N$ (see, e.g., [4], p. 64).

We are now ready to prove Theorem 2:

Proof of Theorem 2. First note that it is enough to show it for $p = 3$. For if α is a $(3, 3)$-form in $\mathbb{C}^6$ such that $\alpha \geqslant 0$ and $\alpha^2 < 0$ then for $p > 3$ we set:

$$\beta := i\,dz_7 \wedge d\bar{z}_7 + \cdots + i dz_{2p} \wedge d\bar{z}_{2p}.$$

We now have $\alpha \wedge \beta^{p-3} \geqslant 0$ but $(\alpha \wedge \beta^{p-3})^2 = \alpha^2 \wedge \beta^{2p-6} < 0$.

Set $p = 3$, so that $N = 20$, and order $\mathcal{I} = \{J_1, \ldots, J_{20}\}$ lexicographically. Then the image of the Plücker embedding (12) is in particular contained in the quadric:

$$z_1 z_{20} - z_{10} z_{11} + z_5 z_{16} - z_2 z_{19} = 0. \tag{13}$$

For positive a, λ, μ to be determined later define:

$$\alpha := i\left[\lambda(dz_{J_1} \wedge d\bar{z}_{J_1} + dz_{J_{20}} \wedge d\bar{z}_{J_{20}}) + \mu \sum_{\substack{k \in \{2,5,10 \\ 11,16,19\}}} dz_{J_k} \wedge d\bar{z}_{J_k} + a(dz_{J_1} \wedge d\bar{z}_{J_{20}} + dz_{J_{20}} \wedge d\bar{z}_{J_1})\right].$$

Then, similarly as in the proof of Proposition 4,

$$\begin{aligned} \bar{z} A z^T &= \lambda\big(|z_1|^2 + |z_{20}|^2\big) + \mu\big(|z_2|^2 + |z_5|^2 + |z_{10}|^2 + |z_{11}|^2 + |z_{16}|^2 + |z_{19}|^2\big) + 2a\,\mathrm{Re}(\bar{z}_1 z_{20}) \\ &\geqslant 2(\lambda - a)|z_1 z_{20}| + 2\mu|-z_{10} z_{11} + z_5 z_{16} - z_2 z_{19}| \\ &= 2(\lambda + \mu - a)|z_1 z_{20}| \end{aligned}$$

if z satisfies (13). Therefore $\alpha \geqslant 0$ if $a \leqslant \lambda + \mu$.

On the other hand, by (11) and (9):

$$\alpha^2 = 2\big(\lambda^2 + 3\mu^2 - a^2\big)\,dV.$$

We see that if we take $a = \lambda + \mu$ and $\lambda > \mu > 0$, then $\alpha \geqslant 0$ but $\alpha^2 < 0$. □

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