C. R. Acad. Sci. Paris, Ser. I 350 (2012) 465–467

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex Analysis/Functional Analysis

On the singular factor of a linear combination of holomorphic functions

Sur le facteur singulier d'une combinaison linéaire de fonctions holomorphes

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ARTICLE INFO

Article history:
Received 24 September 2011
Accepted 11 May 2012
Available online 16 May 2012

Presented by the Editorial Board

ABSTRACT

We prove that the linear combinations of functions $f_0, \dots, f_n \in H^\infty$ have "few" singular inner factors, provided that the f_j's are suitably smooth up to the boundary, while in general this is no longer true.

RÉSUMÉ

On démontre que les combinaisons linéaires des fonctions $f_0, \dots, f_n \in H^\infty$ possèdent « peu » de facteurs singuliers, à condition que les f_j soient suffisamment lisses jusqu'au bord, mais que ceci n'est pas vrai dans le cas général.

1. Introduction

The classical uniqueness theorem tells us that, given a non-null holomorphic function f on a domain $\Omega \subset \mathbb{C}$, the zero set of f is discrete (i.e., has no accumulation points in Ω). This result admits an amusing extension to linear combinations of several functions.

Theorem A. *Suppose $f_0, \dots, f_n$ are linearly independent holomorphic functions on a domain $\Omega \subset \mathbb{C}$. Then there is a discrete subset $\mathcal{E}$ of Ω with the following property: whenever g is a nontrivial linear combination of $f_0, \dots, f_n$, the zeros of g whose multiplicity exceeds n are all contained in $\mathcal{E}$.*

Thus, "deep" zeros (i.e., those of multiplicity greater than n) are forbidden for a non-null linear combination $\sum_{j=0}^{n} \lambda_j f_j$ except on a "thin" set, which depends only on the f_j's but not on the λ_j's. Needless to say, the assumption on the multiplicities can be neither dropped nor relaxed. Furthermore, the exceptional "thin" sets that arise are the same as those in the classical uniqueness theorem, which corresponds to the case $n = 0$.

We strongly suspect that Theorem A must be known. However, having found no appropriate reference, we now give a quick proof.

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[1] Supported in part by grant MTM2011-27932-C02-01 from El Ministerio de Ciencia e Innovación (Spain) and grant 2009-SGR-1303 from AGAUR (Generalitat de Catalunya).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.05.004

Let $W = W(f_0,\dots,f_n)$ denote the *Wronskian* of the functions f_j, so that

$$W(f_0,\dots,f_n) := \begin{vmatrix} f_0 & f_1 & \dots & f_n \\ f_0' & f_1' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_0^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{vmatrix}. \tag{1}$$

The f_j's being linearly independent *and holomorphic*, it follows that W is non-null (and also holomorphic, of course), so its zeros form a discrete subset, say $\mathcal{E}$, of Ω. Now, a point $z \in \Omega$ will be a zero of multiplicity at least $n+1$ for $\sum_{j=0}^n \lambda_j f_j$ if and only if

$$\sum_{j=0}^n \lambda_j f_j^{(k)}(z) = 0 \quad (k = 0,\dots,n). \tag{2}$$

The condition for (2) to have a nontrivial solution $(\lambda_0,\dots,\lambda_n)$ is that the Wronskian matrix $\{f_j^{(k)}(z)\}_{j,k=0}^n$ be singular, i.e., that $W(z) = 0$. The points z in question are, therefore, precisely those in $\mathcal{E}$.

This proof actually shows that the "deep" zeros of *all* the nontrivial linear combinations as above coincide with the zeros of a *single* holomorphic function, namely, of W.

In what follows, the domain Ω will be the disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$. We shall be concerned with functions in the *Hardy spaces* $H^p = H^p(\mathbb{D})$ (see, e.g., [7]) and in the *Hardy–Sobolev spaces*

$$H_k^p := \{f \in H^p: f^{(k)} \in H^p\},$$

where $p > 0$ and k is a positive integer. We remark that if the functions $f_0,\dots,f_n$ from Theorem A are taken to be in H_n^p, then their Wronskian W is in H^q with a suitable q (in case $p \geqslant 1$, this is true with $q = p$). The zero set $\mathcal{E} = W^{-1}(0)$ therefore satisfies the Blaschke condition $\sum_{z\in\mathcal{E}}(1-|z|) < \infty$, and we may rephrase Theorem A as saying that there exists a Blaschke product (the one built from $\mathcal{E}$) with a certain divisibility property.

The purpose of this note is to prove an analog of the above result for *singular inner factors*. Recall that any nontrivial function $f \in H^p$ can be factored canonically as $f = FBS$, where F is an outer function, B a Blaschke product, and S a singular inner function (see [7, Chapter II]). The last-mentioned factor is of the form

$$S(z) = S_\mu(z) := \exp\left\{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,\mathrm{d}\mu(\zeta)\right\},$$

where μ is a (positive) singular measure on the circle $\mathbb{T} := \partial\mathbb{D}$. While the Blaschke product B is determined by the zeros of f in $\mathbb{D}$, the role of the singular factor S is not so easily describable. In a sense, however, such factors can be thought of as responsible for the "boundary zeros of infinite multiplicity".

We prove then, in the spirit of Theorem A, that if $f_0,\dots,f_n$ satisfy the hypothesis of that theorem (with $\Omega = \mathbb{D}$) and are suitably smooth up to $\mathbb{T}$, then there is a *single* singular inner function S divisible by the singular factor of each nontrivial linear combination $\sum_{j=0}^n \lambda_j f_j$. This means that the totality of singular factors resulting from such linear combinations is rather poor. On the other hand, we give an example to the effect that the smoothness assumption is indispensable; in fact, it is not enough to assume that the f_j's are merely in H^∞.

2. Main result

Theorem 2.1. *Let $f_0,\dots,f_n$ be linearly independent functions in H_n^1. Then there is a singular inner function S with the following property: whenever $\lambda_0,\dots,\lambda_n$ are complex numbers with $\sum_{j=0}^n |\lambda_j| > 0$, the singular factor of $\sum_{j=0}^n \lambda_j f_j$ divides S.*

Proof. Let

$$g = \sum_{j=0}^n \lambda_j f_j \tag{3}$$

be a nontrivial linear combination of the f_j's, so that at least one of the coefficients (say, λ_k) is nonzero. Recalling the notation (1) for the Wronskian, put

$$W := W(f_0,\dots,f_n)$$

and

$$W_k := W(f_0,\dots,f_{k-1},g,f_{k+1},\dots,f_n).$$

It should be noted that $W \not\equiv 0$, because the f_j's are linearly independent holomorphic functions, and also that $W \in H^1$. To verify the latter claim, expand the determinant (1) along its last row and observe that the derivatives $f_j^{(k)}$ with $0 \leqslant k \leqslant n-1$ are all in H^∞. Furthermore, it follows from (3) that $W_k = \lambda_k W$. In particular, the inner factors of W and W_k are identical.

Now, if S_g is the singular factor of g, then the inner factors of $g', g'', \dots, g^{(n)}$ are all divisible by S_g. (Indeed, it is known that for every $h \in H_1^1$, the singular factor of h divides that of h'; see [1, Theorem 1] or [2, Lemma 2].) Expanding the determinant W_k along its kth column, $(g, g', \dots, g^{(n)})^T$, and noting that the corresponding cofactors are in H^1, we conclude that S_g divides the inner factor of W_k, or equivalently, of W. The function $S = S_W$, defined as the singular factor of W, has therefore the required property. □

3. An example

We borrow an idea from [2]. Let θ be a nonconstant inner function that omits an uncountable set of values $E \subset \mathbb{D}$. (The existence of such a function with values in $\mathbb{D} \setminus E$, for any prescribed closed set E of zero logarithmic capacity, is established in [3, Chapter 2].) For each $\alpha \in E$, one has

$$\theta - \alpha = S_\alpha \cdot (1 - \bar{\alpha}\theta), \tag{4}$$

with

$$S_\alpha := \frac{\theta - \alpha}{1 - \bar{\alpha}\theta}.$$

Here, S_α is a singular inner function (because α is not in the range of θ), while the other factor in (4) is outer.

Write μ_α for the singular measure associated with S_α. For μ_α-almost every $\zeta \in \mathbb{T}$, we have $S_\alpha(z) \to 0$, and hence $\theta(z) \to \alpha$, as $z \to \zeta$ nontangentially; see [7, Chapter II]. It follows that the supports of μ_α's, with $\alpha \in E$, are pairwise disjoint. The set E being uncountable and the measures μ_α nonzero, we readily deduce that no finite Borel measure μ on $\mathbb{T}$ can satisfy $\mu \geqslant \mu_\alpha$ for all $\alpha \in E$. Consequently, no singular inner function is divisible by every S_α.

Since S_α is the singular factor of $\theta - \alpha$, which is a linear combination of $f_0 = \theta$ and $f_1 = 1$, we see that these f_j's violate the conclusion of Theorem 2.1 with $n = 1$.

4. Concluding remarks

(1) It would be interesting to know if the space H_n^1 in Theorem 2.1 can be replaced by a larger smoothness class (say, by H_n^p with a $p < 1$), and moreover, to find the optimal smoothness condition on the functions f_j.

(2) The author's recent papers [4–6] contain some related work, where smoothness properties of the Wronskian play a role when proving various extensions of the Mason–Stothers *abc* theorem.

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