C. R. Acad. Sci. Paris, Ser. I 350 (2012) 263–266

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional Analysis

# Operators with normal Aluthge transforms ☆

## *Opérateurs et transformations normales de Aluthge*

Ali Oloomi [a], Mehdi Radjabalipour [b,c]

[a] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Kerman, Iran*
[b] *SBUK Center for Linear Algebra and Optimization, University of Kerman, Iran*
[c] *Iranian Academy of Sciences, Tehran, Iran*

ARTICLE INFO

*Article history:*
Received 31 December 2011
Accepted after revision 9 February 2012
Available online 23 February 2012

Presented by the Editorial Board

ABSTRACT

The main purpose of the Note is to show that if the second Aluthge transform of an invertible operator is normal, so it is its first Aluthge transform. This extends results due to Moslehian and Nabavi Sales [Some conditions implying normality of operators, C. R. Math. Acad. Sci. Paris, Ser. I 349 (2011) 251–254] and Rose and Spitkovsky [On the stabilization of the Aluthge sequence, International Journal of Information and Systems Sciences 4 (1) (2008) 178–189]. Also, the structure of an injective operator with normal Aluthge transform is studied.



RÉSUMÉ

Dans cette Note on démontre que, si la deuxième transformation de Aluthge d'un opérateur inversible est normale, alors sa première transformation de Aluthge est aussi normale, on étend ainsi les résultats de Moslehian et Nabavi Sales [Some conditions implying normality of operators, CRAS, Paris, Ser. I 349 (2011) 251–254], et Rose et Spitkovsky [On the stabilization of of the Aluthge sequence, International Journal of Information and Systems Sciences 4 (1) (2008) 178–189]. Par ailleurs on établit la structure d'opérateur injectif avec transformation normale de Altuthge.



## 1. Introduction

For a bounded operator $S$ on a general Hilbert space $K$, its Aluthge transform $\tilde{S}$ is defined as $\tilde{S} := A_S U_S A_S$, where $A_S := |S|^{1/2}$ and $U_S$ is the minimal partial isometry establishing the polar decomposition $S = U_S|S|$ [1]. The Aluthge transform is designed as a measure for the normality of an operator; this is justified by the facts that (i) a normal operator is clearly equal to its Aluthge transform and, in fact, $T = \tilde{T}$ if and only if $T$ is quasi-normal, (ii) the iterated sequence of Aluthge transforms of an $n \times n$ matrix converges to a normal matrix, (iii) if the limit of the iterated sequence of a general bounded operator exists, then the limit is a normal matrix, (iv) if $S$ acts bijectively on a finite dimensional space and if its second Aluthge transform is normal, so it is its first Aluthge transform, and (v) if $\sigma(U_S)$ lies in an open semicircle and if $\tilde{S}$ is normal, then $S$ is normal. It is also known that $\sigma(T) = \sigma(\widetilde{T})$ for all bounded linear operators $T$. (See [2,3,5–7,9].)

☆ The research is supported by the Iranian National Science Foundation.

*E-mail addresses:* ali_oloomi111@yahoo.com (A. Oloomi), mradjabalipour@gmail.com, radjabalipour@ias.ac.ir (M. Radjabalipour).

1631-073X/$ – see front matter © 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2012.02.003

The normality results of [7] and [9] will be extended in the present Note. The structure of injective bounded operators with normal Aluthge transforms is studied in Section 2; the normality of the Aluthge transform of an invertible operator whose second Aluthge transform is normal will be given in Section 3.

## 2. The structure of operators with normal Aluthge transforms

The (minimal) partial isometry $U_S$ of a polar decomposition $S = U_S|S|$ is isometry if and only if $S$ is injective; moreover, $U_S$ is unitary if and only if $S$ is a quasi-affinity. (A quasi-affinity is an operator which is injective and has a dense range.) Also, $S$ and $|S|$ have the same null spaces. In this section, we study the class of all injective operators whose Aluthge transforms are normal. To simplify the arguments, we assume without loss of generality that the underlying Hilbert space is separable; in fact, if $x \in K$ is an arbitrary vector, then the closed linear span $H$ of all words in $S$ and $S^*$ applied to $x$ is a reducing separable invariant subspace of $S$, $S = S_1 \oplus S_2$ and $\tilde{S} = \tilde{S}_1 \oplus \tilde{S}_2$ with respect to $K = H \oplus H^\perp$. Moreover, an operator is normal if and only if its restriction to any reducing separable invariant subspace is normal. Motivated by this observation, we may and shall assume without loss of generality that our underlying Hilbert spaces are separable. Thus, throughout the remainder of the paper, the symbol $T$ will be reserved for a bounded operator on a separable Hilbert space $H$ and the subscript $T$ in $A_T$ and $U_T$ will be dropped. The domain, the null space and the range of a general linear map $S$ are denoted by $\mathcal{D}(S)$, $\ker S$ and $\mathcal{R}(S)$, respectively. It is known that [9] if $T$ is boundedly invertible and $\tilde{T}$ $(= AUA)$ is normal, then $A$ and $U^2$ commute. The following theorem extends this result to the case that $T$ is merely an injective bounded linear operator with a normal Aluthge transform. The theorem will also illustrate a direct integral structure for such operators. (For the definition and properties of direct integrals, see pp. 496–504 of [8] and Theorem 18.1 of [4].) Recall that, for every normal operator $N$ on $H$, $N = N^+ \oplus (-N^-)$ with respect to $H = H^+ \oplus H^-$ in which $H^\pm = \mathcal{R}(E(\Omega^\pm))$, where $E$ is the spectral measure corresponding to $N$,

$$\Omega^+ = \{re^{i\theta}\colon r \geqslant 0; 0 \leqslant \theta < \pi\}, \quad \text{and} \quad \Omega^- = \mathbb{C} \setminus \Omega^+.$$

Clearly,

$$\sigma(N^\pm) \subset \overline{\Omega^+}, \qquad E^+((-\infty, 0)) = 0, \quad \text{and} \quad E^-((-\infty, 0]) = 0,$$

where $E^\pm$ is the spectral measure corresponding to $N^\pm$.

**Theorem 2.1.** *Let $T$ be injective and assume $\tilde{T}$ is normal. Then $U$ is unitary, $U^2 A = AU^2$ and*

$$T = \int_{[0,\pi]}^{\oplus} e^{i\theta} U(\theta) A^2(\theta)\, \mathrm{d}\mu(\theta)$$

*for some positive Borel measure $\mu$ on $[0, \pi]$, where $U(\theta)$ is a unipotent self-adjoint operator and $A(\theta)$ is a positive operator for almost all $\theta$ $[\mu]$.*

**Proof.** Since $T$ is injective, $|T|^\alpha$ is injective for all $\alpha \geqslant 0$ and hence $\overline{\mathcal{R}(|T|^\alpha)} = \ker(|T|^\alpha)^\perp = H$ for all $\alpha \geqslant 0$. Thus, $U$ is an isometry from $H$ onto $\overline{\mathcal{R}(T)}$. Now, since $AUA$ is normal, it follows that $AUA^2U^*A = AU^*A^2UA$ and, hence, $UA^2U^* = U^*A^2U$. (Note that $A$ is an injective operator with a dense range, and all the involving operators are uniformly continuous.) To show $U$ is unitary, it is sufficient to prove that $U$ has a dense range or, equivalently, $\ker U^* = \{0\}$. Let $x \in \ker U^*$ be arbitrary. Then

$$\|AUx\|^2 = \langle U^*A^2Ux, x\rangle = \langle UA^2U^*x, x\rangle = 0,$$

which implies that $AUx = 0$. Since $AU$ is injective, it follows that $x = 0$. Thus, $U^*$ is injective and, hence, unitary. Moreover, $U^2A^2 = A^2U^2$ and, by positivity of $A$, $U^2A^\alpha = A^\alpha U^2$ for all $\alpha \geqslant 0$.

Now that $U$ is unitary (and, hence, normal), consider the direct sum $U = U^+ \oplus (-U^-)$ with the underlying direct sum $H = H^+ \oplus H^-$. Let $E$, $F$ and $G$ be the spectral measures corresponding to $U$, $U^2$ and $U^+ \oplus U^-$, respectively. Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = z^2$. Then, for every Borel subset $\Delta$ of unit circle $\mathbb{T}$,

$$F(\Delta) = \chi_\Delta(U^2) = (\chi_\Delta o g)(U) = \chi_{g^{-1}(\Delta)}(U) = E(g^{-1}(\Delta)).$$

Letting $\Gamma = \Omega^+ \cap g^{-1}(\Delta)$ yields

$$E(g^{-1}(\Delta)) = E(\Gamma \cup (-\Gamma)) = E^+(\Gamma) \oplus E^-(\Gamma) = G(\Gamma),$$

which implies that the commutant of $U^+ \oplus U^-$ is a subset of the commutant of $U^2$. Conversely, if $\Gamma \subset \mathbb{T}$ and $\Gamma_0 = \Gamma \cap \Omega^+$, then

$$G(\Gamma) = G(\Gamma_0) = E^+(\Gamma_0) \oplus E^-(\Gamma_0) = E(\Gamma_0 \cup -\Gamma_0) = E(g^{-1}(g(\Gamma_0))) = F(g(\Gamma_0)),$$

which implies that the commutant of $U^2$ is a subset of and, hence, equal to the commutant of $U^+ \oplus U^-$.

Next, define $h(e^{i\theta})=\theta$ for $0\leqslant\theta\leqslant\pi$ and let $B^{\pm}=h(U^{\pm})$. Note that $B^+\oplus B^-=h(U^+\oplus U^-)$ and, hence, the commutant of $B^+\oplus B^-$ is equal to that of $U^+\oplus U^-$. By [8, pp. 496–504], there exist positive Borel measures $\mu^{\pm}$ on $[0,\pi]$ such that

$$H^{\pm}=\int^{\oplus} H^{\pm}(\theta)\,\mathrm{d}\mu^{\pm}(\theta), \quad \text{and} \quad B^{\pm}=\int^{\oplus} B^{\pm}(\theta)\,\mathrm{d}\mu^{\pm}(\theta),$$

for appropriate families of Hilbert spaces $\{H^{\pm}(\theta)\}$ and positive operators $\{B^{\pm}(\theta)\}$. Letting $\mu=\mu^++\mu^-$ and $w^{\pm}(\theta)=\mathrm{d}\mu^{\pm}/\mathrm{d}\mu$, one can identify $\int^{\oplus} H^+(\theta)\,\mathrm{d}\mu(\theta)\oplus\int^{\oplus} H^-(\theta)\,\mathrm{d}\mu(\theta)$ and $\int^{\oplus}(H^+(\theta)\oplus H^-(\theta))\,\mathrm{d}\mu(\theta)$ via the unitary transformation

$$\int^{\oplus} f^+(\theta)\,\mathrm{d}\mu^+(\theta)\oplus\int^{\oplus} f^-(\theta)\,\mathrm{d}\mu^-(\theta)\mapsto\int^{\oplus}\left\{f^+(\theta)\left[w^+(\theta)\right]^{1/2}\oplus f^-(\theta)\left[w^-(\theta)\right]^{1/2}\right\}\mathrm{d}\mu(\theta).$$

Moreover, up to unitary equivalence,

$$B^+\oplus\pm B^-=\int^{\oplus}\theta\begin{bmatrix} I & 0\\ 0 & \pm I\end{bmatrix}\mathrm{d}\mu(\theta), \tag{1}$$

$$U^+\oplus\pm U^-=\int^{\oplus}e^{i\theta}\begin{bmatrix} I & 0\\ 0 & \pm I\end{bmatrix}\mathrm{d}\mu(\theta), \quad \text{and} \tag{2}$$

$$|T|=A^2=\int^{\oplus}\begin{bmatrix} A_{11}(\theta) & A_{12}(\theta)\\ A_{12}^*(\theta) & A_{22}(\theta)\end{bmatrix}\mathrm{d}\mu(\theta), \tag{3}$$

where the integrand of the last equation is positive definite for almost all $\theta[\mu]$. □

The following corollary generalizes Theorem 2.5 of [7]:

**Corollary 2.2.** *Assume $T$ is an injective operator and $\tilde{T}$ is normal. Then the following assertions are true*:

(1) *$T=T_1\oplus T_2$, where $T_1$ is the maximal reducing normal part of $T$.*
(2) *For every Borel subset $\Delta$ of the unit circle $\mathbb{T}$, $E_2(-\Delta)=0$ whenever $E_2(\Delta)=0$, where $E_2$ is the spectral measure corresponding to $T_2$.*

**Proof.** The proof of the existence of $T_1$ is a simple maximality argument. So, we assume without loss of generality that $T=T_2$. Let $\Delta$ be a Borel subset of $\mathbb{T}$ and assume $E(\Delta)=0$. Let $\Lambda=h(\Delta\cap\Omega^+)$. By the proof of Theorem 2.1, $H^+(\theta)=\{0\}$ and, hence, $A^2(\theta)=A_{22}(\theta)$ for almost all $\theta\in\Lambda$. Thus $A^2(\theta)U(\theta)=U(\theta)A^2(\theta)$, which implies that the subspace $\int_{\Lambda}^{\oplus} H(\theta)\,\mathrm{d}\mu(\theta)$ is a reducing invariant subspace of $T$, on which $T$ reduces to a normal part. Thus $\int_{\Lambda}^{\oplus}\Lambda H(\theta)\,\mathrm{d}\mu(\theta)=\{0\}$ and $E(-\Delta)=0$. A similar argument applied to $\Delta\cap\Omega^-$ finishes the proof. □

The proof of Theorem 2.1 suggests the proof of the following proposition:

**Proposition 2.3.** *For a normal quasi-affinity $N$, the commutant of $N^2$ is equal to the commutant of $N^+\oplus N^-$.*

## 3. Invertible operators with normal second Aluthge transforms

In this section, we show that if $T$ is the Aluthge transform of an invertible operator $S$ and if $\tilde{T}$ is normal, then $T$ is normal. The proof of the finite dimensional case in [9] has greatly motivated the proof of the infinite dimensional case given below.

**Theorem 3.1.** *Let $T=\tilde{S}$ for some invertible operator $S$ and assume $\tilde{T}$ is normal. Then $T$ is normal.*

**Proof.** Note that an invertible operator is similar to its Aluthge transform. Thus $T$, $S$ and $\tilde{T}$ are all invertible. Also, $T(T^*)^{-1}=A_SU_S^2A_S^{-1}$, which implies that the spectrum of $T(T^*)^{-1}$ lies on the unit circle. Now, since $\tilde{T}$ is normal, it follows that $T$, $T^*$ and $T^{-1}$ all lie in the commutant of $U^2$ and, by Theorem 2.1,

$$T\left(T^*\right)^{-1}=\int^{\oplus}e^{2i\theta}U(\theta)A(\theta)U(\theta)A(\theta)^{-1}\,\mathrm{d}\mu(\theta)=R\int^{\oplus}C(\theta)\,\mathrm{d}\mu(\theta)\,R^{-1},$$

where $R=\int^{\oplus}A(\theta)^{1/2}\,\mathrm{d}\mu(\theta)$ and

$$C(\theta)=e^{2i\theta}A(\theta)^{-1/2}U(\theta)A(\theta)U(\theta)A(\theta)^{-1/2}. \tag{4}$$

Thus the spectrum of $C(\theta)$ lies on the unit circle a.e. $[\mu]$. Since $e^{-2i\theta}C(\theta)$ is positive, its spectrum is equal to $\{1\}$ and, hence, $C(\theta)=e^{2i\theta}I$ a.e. $[\mu]$. Therefore, $A(\theta)^{1/2}U(\theta)A(\theta)^{-1/2}=A(\theta)^{-1/2}U(\theta)A(\theta)^{1/2}$ and, thus $A(\theta)U(\theta)=U(\theta)A(\theta)$ a.e. $[\mu]$. Therefore, $|T|$ and $U$ commute, which means that $T$ is normal. □

**Remark 3.2.** Let $M = \ker A^{\perp}$ and let $P : H \to H$ be the orthogonal projection onto $M$. Assume $B = A|_M$, $V = PU|_M$ and $W = (I - P)U|_M$. It is well known that $\tilde{T}^j = 0 \oplus (BVB)^j$ and

$$T^j = UA\tilde{T}^{j-1}A = \begin{bmatrix} 0 & WB(BVB)^{j-1}B \\ 0 & VB(BVB)^{j-1}B \end{bmatrix}$$

for $j = 1, 2, \ldots$. Thus $T$ is a nilpotent operator of order $k$ if and only if $\tilde{T}$ is a nilpotent operator of order $k - 1$. This reveals that the second Aluthge transform of a singular operator may be normal but its first Aluthge transform be non-normal. Theorem 3.1 shows that this cannot happen for invertible operators. However, Theorem 2.1 leaves the following question open:

**Question.** Assume the second Aluthge transform of a quasi-affinity is normal. Is it always true that its first Aluthge transform is normal?

## References

[1] A. Aluthge, On $p$-hyponormal operators for $0 < p < 1$, Integral Equations and Operator Theory 13 (3) (1990) 307–315.
[2] T. Ando, Aluthge transforms and the convex hull of the eigenvalues of a matrix, Linear and Multilinear Algebra 52 (2004) 281–292.
[3] J. Antezana, E.R. Pujlas, D. Stojanoff, The iterated Aluthge transforms of a matrix converge, Advances in Mathematics 226 (2) (2011) 1591–1620.
[4] M.S. Brodskiĭ, Triangular and Jordan Representations of Linear Operators, Nauka, Moscow, 1969; English translation: Translations of Mathematical Monographs, vol. 32, Amer. Math. Soc., Providence, RI, 1971.
[5] I.B. Jung, E. Ko, C. Pearcy, Aluthge transform of operators, Integral Equations and Operator Theory 37 (2000) 437–448.
[6] I.B. Jung, E. Ko, C. Pearcy, The iterated Aluthge transform of an operator, Integral Equations and Operator Theory 45 (4) (2003) 375–387.
[7] M.S. Moslehian, S.M.S. Nabavi Sales, Some conditions implying normality of operators, C. R. Math. Acad. Sci. Paris, Ser. I 349 (2011) 251–254.
[8] M.A. Naĭmark, Normed Rings, GITTL, Moscow, 1956; English translation: Noordhoff, Groningen, 1959.
[9] D.E.V. Rose, I.M. Spitkovsky, On the stabilization of the Aluthge sequence, International Journal of Information and Systems Sciences 4 (1) (2008) 178–189.