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Geometry

Vector bundles on toric varieties ☆

Fibrés vectoriels sur les variétés toriques

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ABSTRACT

Following Sam Payne's work, we study the existence problem of nontrivial vector bundles on toric varieties. The first result we prove is that every complete fan admits a nontrivial conewise linear multivalued function. Such functions could potentially be the Chern classes of toric vector bundles. Then we use the results of Cortiñas, Haesemeyer, Walker and Weibel to show that the (non-equivariant) Grothendieck group of the toric 3-fold studied by Payne is large, so the variety has a nontrivial vector bundle. Using the same computation, we show that every toric 3-fold X either has a nontrivial line bundle, or there is a finite surjective toric morphism from Y to X, such that Y has a large Grothendieck group.

RÉSUMÉ

Suivant un travail de Sam Payne nous étudions l'existence de fibrés vectoriels non triviaux sur une variété torique. Notre premier résultat établit que tout éventail complet admet une fonction, non triviale, qui est linéaire et multi-valuée sur chaque cône. Une telle fonction peut potentiellement être la classe de Chern d'un fibré vectoriel torique. Nous utilisons alors un résultat de Cortiñas, Haesemeyer, Walker et Weibel pour montrer que le groupe de Grothendieck (non équivariant) de la variété torique de dimension 3 étudiée par Payne est grand et ainsi la variété a un fibré vectoriel non trivial. Par un calcul similaire nous montrons que pour toute variété torique X de dimension 3, soit X a un fibré en droites non trivial, soit il existe un morphisme torique, surjectif, fini de Y sur X, où Y a un grand groupe de Grothendieck.

1. Introduction

The purpose of this Note is to study the existence of nontrivial vector bundles on toric varieties. Sam Payne in [9] raised the question whether every complete toric variety has a nontrivial vector bundle. He gave an example of a 3-dimensional toric variety that does not admit a nontrivial equivariant vector bundle of rank at most 3. His proof was based on studying the equivariant Chern classes of torus-equivariant bundles (toric bundles for short). Since a toric vector bundle splits on a torus invariant affine open subset, its equivariant total Chern class also splits on such open sets. Payne proved that the

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particular toric 3-fold does not admit any nontrivial equivariant Chern classes of degree 3 or less, and toric vector bundles of rank 3 or less with trivial equivariant total Chern class are all trivial.

Our first result is that one cannot use Payne's method to prove nonexistence of toric vector bundles in arbitrary rank. We show that every complete toric variety has a nontrivial equivariant Chern class of sufficiently high degree. (By a Chern class we mean an equivariant cohomology class that splits on every torus invariant affine open, hence is a candidate for the total equivariant Chern class of a toric vector bundle.)

The second result we prove is an existence result of non-equivariant vector bundles using the non-equivariant algebraic K-theory of toric varieties. The K-theory of smooth toric varieties is well understood [7,10,6]. Singular toric varieties, on the other hand, can have very complicated and large K-groups. Joseph Gubeladze in [4] gave an example of a toric variety X such that $K_0(X)$ is as large as the ground field k. Thus, if k is uncountable, then the K-group has an uncountable rank. Another such example was given by Cortiñas, Haesemeyer, Walker and Weibel in [2]. We will use the results of [2] to show that the toric 3-fold studied by Payne is of the same type: its K-group has an uncountable rank. This in particular shows that the 3-fold has nontrivial vector bundles of high enough rank.

Using the same K-group computation, we also show that every complete toric 3-fold X either has a nontrivial line bundle, or there exists a finite surjective toric morphism $Y \to X$ such that $K_0(Y)$ has an uncountable rank, and in particular, admits a nontrivial vector bundle.

Forgetting the equivariant structure of a vector bundle induces a homomorphism from equivariant to ordinary K-group $K_0^T(X) \to K_0(X)$. This homomorphism may not be surjective when X is singular, hence $K_0^T(X)$ could be zero, even though $K_0(X)$ is large. For example, we do not know if Payne's 3-fold has any nontrivial equivariant vector bundles, even though it has lots of non-equivariant ones.

2. Equivariant Chern classes of toric vector bundles

We work over an algebraically closed field k of characteristic zero. We will assume that k is uncountable. This is needed below to claim that some K-group has an uncountable rank.

We refer to Fulton [3] for notation on toric varieties. Let N be a lattice and M its dual lattice. A toric variety is determined by a pair (Δ, N), where Δ is a fan in the lattice N. The set of one-dimensional cones of Δ is denoted $\Delta(1)$. For $\rho \in \Delta(1)$, n_ρ is the first nonzero lattice point on ρ. The affine toric variety associated to a cone σ in N is U_σ, the big torus is T, and the divisor corresponding to $\rho \in \Delta(1)$ is denoted V_ρ.

We refer to [9] for background material on toric vector bundles. A toric vector bundle is a torus-equivariant vector bundle on a toric variety. Such bundles were classified by Klyachko [5].

Let X_Δ be a toric variety corresponding to a fan Δ. Equivariant Chern classes of toric vector bundles lie in the equivariant Chow cohomology ring $A_T^*(X_\Delta)$, which can be identified with the ring of integral conewise polynomial functions on the fan Δ [1,8]. A toric line bundle on X_Δ is determined by its equivariant first Chern class, which is a conewise linear integral function on the fan Δ. A toric vector bundle splits into a direct sum of line bundles on every affine open $U_\sigma \subset X_\Delta$ for $\sigma \in \Delta$. This splitting gives an r-element multiset L_σ of integral linear functions on every cone σ. When τ is a face of σ, then the multiset L_σ restricts to the multiset L_τ. The collection $\{L_\sigma\}_{\sigma\in\Delta}$ is called a multivalued integral conewise linear function on Δ. Its degree is $r = |L_\sigma|$. A multivalued conewise linear function is called trivial if there exists a multiset L of global linear functions, such that L_σ is the restriction of L to σ for all cones $\sigma \in \Delta$.

Given a symmetric polynomial $s(x_1,\dots,x_r)$ in r variables and a multivalued conewise linear function $\{L_\sigma\}_{\sigma\in\Delta}$ of degree r, one can produce a conewise polynomial function by substituting on each cone σ the functions in L_σ into s. When s_i is the i-th elementary symmetric function and $\{L_\sigma\}$ the multivalued conewise linear function associated to a toric vector bundle E, then the conewise polynomial function produced this way is the i-th equivariant Chern class of E. Thus, the multivalued function encodes all the equivariant Chern class data of E.

Theorem 2.1. *Let Δ be a fan (rational, polyhedral) in the lattice N having more than one maximal cone with dimension equal to the rank of N. Then there exists a nontrivial multivalued conewise linear integral function on Δ.*

Lemma 2.2. *For every rational cone σ there exist two different multivalued linear integral functions on it such that the restriction of these two functions on the facets of σ is equal as multivalued linear functions.*

Proof of Lemma 2.2. We may assume that σ has dimension equal to the rank of N. Let $\tau_1,\dots,\tau_k$ be the facets of σ. Pick nonzero elements $L_i \in M \cap \sigma^\vee \cap \tau_i^\perp$. For $j = 1, 2$, consider the 2^{k-1}-element multisets

$$A_j = \left\{\varepsilon_1 L_1 + \cdots + \varepsilon_k L_k \middle| \varepsilon_i \in \{0,1\}, \sum_i \varepsilon_i = j \ (\mathrm{mod}\ 2)\right\}.$$

Then the two multisets A_1, A_2 restrict to the same multiset on any τ_i (to get a bijection between the two restrictions to τ_i, just change the coefficient ε_i). The two multisets A_1, A_2 are not equal; for example, 0 lies in A_2, but not in A_1. □

Proof of the Theorem 2.1. Pick an arbitrary cone $\sigma \in \Delta$ of maximal dimension. By the lemma, there are two distinct multivalued linear functions A_1, A_2 on σ that restrict to the same multivalued function on every face τ of σ. To define the conewise linear function L on Δ, set L equal to A_1 on σ and to A_2 outside of σ. This function is nontrivial, i.e., not a multiset of global linear functions, because there exists another cone σ' of maximal dimension on which the two multisets A_1, A_2 are not equal. □

3. Non-equivariant vector bundles via Grothendieck's *K*-group

We use the results of Cortiñas, Haesemeyer, Walker and Weibel [2] to show that the example 3-fold studied by Payne [9] has a large Grothendieck group, hence it must have nontrivial vector bundles. For general complete toric 3-folds we show that it either has a nontrivial line bundle or else we can construct a finite surjective cover with large Grothendieck group.

Let $X = X_\Delta$ be a toric 3-fold. It is shown in [2] that the rational K-group $K_0(X) \otimes \mathbb{Q}$ contains as a direct summand the cohomology $H^1(X, \mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on X. This sheaf is the cokernel of a morphism $\Omega^1_X \to \tilde{\Omega}^1_X$, where Ω^1_X is the sheaf of regular differential 1-forms on X, $\tilde{\Omega}^1_X$ is the Danilov's sheaf of 1-forms (described below). Since the sheaves are defined over the ground field k, the cohomology group is a k-vector space. Thus, if k is an extension of $\mathbb{Q}$ of uncountable degree and the cohomology group is nonzero, then the rank of $K_0(X)$ must also be uncountable.

Let us first recall the definitions of the sheaf $\tilde{\Omega}^1_X$ and the morphism $\Omega^1_X \to \tilde{\Omega}^1_X$ following [2]. The sheaf $\tilde{\Omega}^1_X$ is defined by the exact sequence

$$0 \to \tilde{\Omega}^1_X \to \mathcal{O}_X \otimes M \xrightarrow{\delta} \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{V_\rho},$$

where δ maps $f \otimes q \in \mathcal{O}_X \otimes M$ to $f|_{V_\rho} \cdot \langle n_\rho, q\rangle$. The sheaf $\mathcal{O}_X \otimes M$ can be identified with the sheaf $\Omega^1_X(\log D)$ of differentials with log poles along $D = X \smallsetminus T$. The identification takes $f \otimes q \in \mathcal{O}_X \otimes M$ to the form $f \cdot \frac{d\chi^q}{\chi^q}$. The morphism $\Omega^1_X \to \tilde{\Omega}^1_X$ sends a form $\chi^p \, d\chi^q \in \Omega^1_X(U_\sigma)$ to $\chi^{p+q} \otimes q \in \mathcal{O}_X(U_\sigma) \otimes M$. Its image lies in $\tilde{\Omega}^1_X$.

The two sheaves $\tilde{\Omega}^1_X$, Ω^1_X and the morphism between them are T-equivariant, hence M-graded on open T-invariant sets U_σ. If $m \in \sigma^\vee \cap M$, $q \in M$, then $\chi^m \otimes q$ is homogeneous of degree m. The sections $\tilde{\Omega}^1_X(U_\sigma)$ of degree m can be described as follows. If $m \notin \sigma^\vee$, then $\tilde{\Omega}^1_X(U_\sigma)_m = 0$. Otherwise, let $\sigma^\vee_m$ be the smallest face of $\sigma^\vee$ containing m. Then

$$\tilde{\Omega}^1_X(U_\sigma)_m = k\chi^m \otimes \big(\operatorname{Span}(\sigma^\vee_m) \cap M\big).$$

Let us identify the image of the map $\Omega^1_X(U_\sigma)_m \to \tilde{\Omega}^1_X(U_\sigma)_m$. If $p, q \in \sigma^\vee \cap M$, $p + q = m$, then $\chi^p \, d\chi^q$ has degree m and maps to $\chi^{p+q} \otimes q$. Thus, the image is

$$k\chi^m \otimes \big(\operatorname{Span}\{q \mid q \in \sigma^\vee \cap M, m - q \in \sigma^\vee \cap M\} \cap M\big) \subset k\chi^m \otimes \big(\operatorname{Span}(\sigma^\vee_m) \cap M\big).$$

A particular case that we need below is when $\operatorname{Span}(\sigma^\vee_m)$ has dimension 1. In that case $\tilde{\Omega}^1_X(U_\sigma)_m$ has dimension 1 and the image of $\Omega^1_X(U_\sigma)_m$ is nonzero (take $q = m$), hence $\Omega^1_X(U_\sigma)_m \to \tilde{\Omega}^1_X(U_\sigma)_m$ is surjective.

3.1. Payne's toric 3-fold

We now turn to the 3-fold in [9]. Let $N = \mathbb{Z}^3$ and start with the fan over the faces of the regular cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. To get Δ, we deform this fan by replacing $(1, -1, 1)$ with $(1, -1, 2)$ and $(1, 1, 1)$ with $(1, 2, 3)$.

Lemma 3.1. *With Δ as above, the group $K_0(X(\Delta))$ has uncountable rank.*

Proof. Let $\tau \in \Delta$ be the 2-dimensional cone generated by $(1, -1, -1)$ and $(1, -1, 2)$, and let σ_1, σ_2 be the two 3-dimensional cones having τ as a face.

First let $X = U_{\sigma_1} \cup U_{\sigma_2}$. Denote $\mathcal{F} = \operatorname{Coker}(\Omega^1_X \to \tilde{\Omega}^1_X)$. Mayer–Vietoris sequence for the cover $X = U_{\sigma_1} \cup U_{\sigma_2}$ is

$$\cdots \to H^0(U_{\sigma_1}, \mathcal{F}) \oplus H^0(U_{\sigma_2}, \mathcal{F}) \to H^0(U_\tau, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U_{\sigma_1}, \mathcal{F}) \oplus H^1(U_{\sigma_2}, \mathcal{F}) \to \cdots.$$

Since U_{σ_i} are affine and $\mathcal{F}$ is coherent, the cohomology groups $H^1(U_{\sigma_i}, \mathcal{F})$ vanish for $i = 1, 2$.

Let us fix $m = (1, -1, 0) \in \mathbb{Z}^3 = M$. We claim that $H^0(U_{\sigma_i}, \mathcal{F})_m = 0$ for $i = 1, 2$, but $H^0(U_\tau, \mathcal{F})_m \neq 0$. This implies that $H^1(X, \mathcal{F}) \neq 0$.

Let σ_1 be the cone generated by $(1, -1, -1), (1, -1, 2), (1, 1, -1), (1, 2, 3)$. Since $\langle m, (1, 2, 3)\rangle < 0$, $m \notin \sigma^\vee$, hence $H^0(U_{\sigma_1}, \mathcal{F})_m = 0$. The other cone σ_2 is generated by $(1, -1, -1), (1, -1, 2), (-1, -1, -1), (-1, -1, 1)$. Now $m \in \sigma_2^\vee$, but it vanishes on the face of σ generated by $(-1, -1, -1), (-1, -1, 1)$, hence $(\sigma_2)^\vee_m$ has dimension 1 and $H^0(U_{\sigma_2}, \mathcal{F})_m = 0$.

The cone $\tau^\vee$ is generated by $(0, -1, 1), (0, -2, -1), \pm(1, 1, 0)$. The semigroup $\tau^\vee \cap M$ is generated by these lattice points and m. The intersection $\tau^\vee \cap (-\tau^\vee + m) \cap M$ spans a plane in $\mathbb{R}^3$. Since m lies in the interior of $\tau^\vee$, $\operatorname{Span} \tau^\vee_m = \mathbb{R}^3$. It follows that $\dim \mathcal{F}(U_\tau)_m = 1$.

This finishes the proof that $H^1(X,\mathcal{F}) \neq 0$ when $X = U_{\sigma_1} \cup U_{\sigma_2}$. Next consider $X_\Delta = X \cup Y$, where Y is the toric variety corresponding to the fan $\Delta \smallsetminus \{\sigma_1, \sigma_2, \tau\}$. Mayer–Vietoris sequence for the open cover is

$$\cdots \to H^1(X_\Delta, \mathcal{F}) \to H^1(X, \mathcal{F}) \oplus H^1(Y, \mathcal{F}) \to H^1(X \cap Y, \mathcal{F}) \to \cdots.$$

We claim that $H^1(X \cap Y, \mathcal{F}) = 0$, which then implies that $H^1(X_\Delta, \mathcal{F}) \neq 0$. Indeed, $X \cap Y$ is covered by open affines U_{τ_i}, where τ_i has dimension 2. The intersections $U_{\tau_i} \cap U_{\tau_j}$ for $i \neq j$ are smooth, hence $\mathcal{F}$ is zero on these intersections. Using Čech cohomology, we see that $H^1(X \cap Y, \mathcal{F}) = 0$. □

3.2. The case of arbitrary complete toric 3-folds

Let X be a toric variety corresponding to a 3-dimensional complete fan Δ. We consider two cases: when every ray in Δ has at least 4 neighbors, and when some ray has only 3 neighbors. In the first case X has a nontrivial line bundle. In the second case there exists a finite surjective toric morphism $X' \to X$, such that $K_0(X')$ is uncountable.

Lemma 3.2. *Suppose every* 1*-dimensional cone* $\rho \in \Delta$ *is contained in at least four* 2*-dimensional cones of* Δ*. Then there exists a nontrivial integral conewise linear function on the fan* Δ*.*

Proof. This is a simple dimension count and left to the reader. □

Lemma 3.3. *Let* $\rho \in \Delta$ *be* 1*-dimensional cone that is contained in only three* 2*-dimensional cones of* Δ*. Then there exists a sublattice of finite index* $N' \subset N$*, such that the toric variety* X' *defined by* (Δ, N') *has* $K_0(X')$ *of uncountable rank.*

Proof. Let τ, τ_1, τ_2 be the 2-dimensional cones containing ρ. By simple geometric considerations one can choose $l \in M$, such that $l \in \mathrm{Rel.Int}\, \tau^\vee$, but $l \notin \tau_1^\vee$, $l \notin \tau_2^\vee$. Replacing N by a sublattice of finite index and l by a multiple $p \cdot l$ if necessary, we may assume that the cone τ is nonsingular with primitive generators v_1, v_2, and $l(v_1) = l(v_2) = 1$. Set $m = l/2$ and $N' = \{v \in N \mid \langle m, v\rangle \in \mathbb{Z}\}$. We now repeat the proof that $H^1(X', \mathcal{F}) \neq 0$ as above, using the lattice point $m \subset M' = \mathrm{Hom}(N', \mathbb{Z})$ and the cone τ. □

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