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Geometry

Abelian fibrations and SYZ mirror conjecture

Fibrations abéliennes et la conjecture « miroir » SYZ

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ABSTRACT

SYZ mirror conjecture predicts that a Calabi–Yau manifold X consists of a family of tori which are dual to a family of special Lagrangian tori on the mirror dual manifold $\widehat{X}$. Here we consider a fibration of polarized abelian varieties and we construct a dual one. Moreover we prove that they are equivalent at the level of derived categories.

RÉSUMÉ

La conjecture de « symétrie miroir SYZ » prédit qu'une variété de Calabi–Yau X consiste en une famille de tores qui sont duaux d'une famille de tores lagrangiennes spéciaux dans la variété miroir duale $\widehat{X}$. Nous considérons ici une fibration de variétés abéliennes polarisées et nous en construisons la duale. De plus, nous montrons qu'elles sont équivalentes au niveau des catégories dérivées.

1. Introduction

A Calabi–Yau space has two kind of moduli spaces, the moduli space of inequivalent complex structures and the moduli space of symplectic structures. Mirror Symmetry should consist in the identification of the moduli space of complex structures on an n-dimensional Calabi–Yau manifold X with the moduli space of complexified Kähler structures on the mirror manifold $\widehat{X}$.

In the case X is an elliptic curve, the modulus of complex structures can be identified with the upper half-plane $\mathbb{H}$ by

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}).$$

We call it X_τ, where τ is the Teichmüller parameter. The second modulus is the kähler class $[w] \in H^2(X, \mathbb{C})$ parametrized by $t \in \mathbb{H}$, as $\int_X w = 2\pi i t$. See [2].

Here we will study Calabi–Yau spaces that are fibred over the same base B by polarized abelian varieties. If X/B is an abelian fibration with a global polarization, and we call $X^\vee/B$ its dual fibration, our main result is:

Theorem 1.1. *The derived categories of both fibrations X/B and $X^\vee/B$ are equivalent.*

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Corollary 1.2. *There is an equivalence $\phi_b : D^b(X_b) \to D^b(\widehat{X}_b)$ for every closed point $b \in B$.*

2. SYZ mirror conjecture

There are two main mathematical conjectures in Mirror Symmetry, Kontsevich homological mirror symmetry conjecture and the conjecture of Strominger, Yau and Zaslow, which predicts the structure of a CY manifold and how to get the mirror of a given CY manifold. We first recall what is a Calabi–Yau manifold.

Definition 2.1. A Calabi–Yau manifold X of dimension n is a smooth compact connected n-fold with vanishing first Betti number and trivial canonical class

$$\Lambda^n \Omega_X \cong K_X \cong \mathcal{O}_X.$$

Let $\pi : X \to S$ be a proper map, when all the fibers are equidimensional, we say that it is a fibration. We don't impose further assumptions on the base. It is of interest from the point of view of Mirror Symmetry, the case in which the total fibration is Calabi–Yau. In this case, Strominger, Yau and Zaslow have conjectured how would it be the structure of the mirror fibration. Mirror dual Calabi–Yau manifolds should be fibred over the same base in such a way that generic fibres are dual tori, and each fibre of any of these two fibrations is a Lagrangian submanifold.

Definition 2.2. Let (X, w) be a holomorphic symplectic manifold (not necessarily compact) of dimension $2r$. A Lagrangian fibration is a proper map $h : X \to B$ onto a manifold B such that the general fibre F of h is Lagrangian, that is, F is connected, of dimension r, and the restriction $w|_F = 0$ vanishes. This implies that the smooth fibres of h are complex tori.

2.0.1. SYZ mirror conjecture

If X and $\widetilde{X}$ are a mirror pair of CY n-folds, then there exists fibrations $f : X \to B$ and $\check{f} : \check{X} \to B$ whose fibres are special Lagrangian with general fibre an n-torus. Furthermore, these fibrations are dual in the sense that canonically $X_b = H^1(\check{X}_b, \mathbb{R}/\mathbb{Z})$ and $\check{X}_b = H^1(X_b, \mathbb{R}/\mathbb{Z})$ whenever X_b and $\check{X}_b$ are non-singular tori. In particular, each of these fibrations admits a canonical section that is an n-cycle having intersection number 1 with the fibre cycle (see [3]). One of the main problems of SYZ mirror conjecture is the presence of singular fibres on the fibration. We will study the moduli problem in this case.

2.1. Abelian fibrations

Let $\Gamma \cong \mathbb{Z}^d$ be a lattice in a complex vector space U of dimension d and $\Gamma^* \subset U^*$ be the dual lattice. The complex torus $(U/\Gamma, I)$ where I is the complex structure, is an abelian variety A of dimension d over $\mathbb{Z}$ if it is algebraic. Let $\widehat{A}$ be the dual abelian variety, i.e. the dual torus $(U^*/\Gamma^*, -I^t)$. There is a unique line bundle P on the product $A \times \widehat{A}$ such that for any point $\alpha \in \widehat{A}$, the restriction P_α on $A \times \{\alpha\}$ represents an element of $Pic^0(A)$ corresponding to α and in addition, the restriction $P|_{\{0\}\times\widehat{A}}$ is trivial. Such P is called the Poincaré line bundle and gives an equivalence between the derived categories of sheaves on A and $\widehat{A}$

2.1.1. The moduli problem of the dual fibration

Let $p : X \to B$ be a fibration by abelian varieties with a relative polarization or ample line bundle defined over the total fibration such that the restriction to each fibre is the polarization class on the corresponding fibre. The existence of a relative polarization for the fibration does not necessarily imply the existence of a section.

In the fibration X/B singular fibres can appear. In this case, except for some particular cases, we don't know how does the dual abelian variety look like. The idea is to replace the abelian variety by one that is derived equivalent to it. We consider the moduli problem of the dual fibration, that is, the dual fibration as the stack representing the Picard functor, that is, the moduli functor of semistable sheaves on the fibres that contains line bundles of degree 0 on smooth fibres. The corresponding coarse moduli space is not a fine moduli space due to the presence of singular fibres. Let us call $X^\vee$ the dual fibration when it exists and satisfying the property that over the smooth locus the fibres correspond to the dual abelian varieties of the original fibration.

Conjecture 2.3. *Two Calabi–Yau threefolds C_1, C_2 that are fibred over the same base $\mathbb{P}^1$ in such a way that the fibres are abelian surfaces and derived equivalent are derived equivalent themselves. This prediction is according with SYZ mirror symmetry conjecture. See [5] and [3].*

First we fix our attention on the smooth locus. Let $\Sigma(p) \hookrightarrow B$ be the discriminant locus of p, that is, the closed subvariety in the parameter space B corresponding to the singular fibres.

$$\begin{array}{lll} X^\vee \supset X^\vee - p^{-1}(\Sigma(p)) & \hookrightarrow & \mathbb{P}^N \\ \downarrow & & \\ B \quad\;\; \supset B - \Sigma(p) & & \end{array} \tag{1}$$

Then, we can take the Zariski closure of $X^\vee - p^{-1}(\Sigma(p))$ in $\mathbb{P}^N$. For each $b \in B - \Sigma(p)$, the corresponding derived equivalence of the fibres X_b and $X_b^\vee$ is given by the Poincare bundle $\mathcal{P}_b$ over the product $X_b \times X_b^\vee$. Moreover, its first Chern class $c_1(\mathcal{P}_b)$ lives in $H^{1,1}(X_b \times X_b^\vee, \mathbb{Z}) \cap H^2(X_b \times X_b^\vee, \mathbb{Z})$. The monodromy group is defined by the action of the fundamental group of the complement of the discriminant locus $\pi_1(B - \Sigma(p))$ on the cohomology $H^*(X_b \times X_b^\vee, \mathbb{Z})$ of a fixed non-singular fiber, and since the class of the polarization is invariant by the monodromy, by Deligne theorem we can extend the class of the Poincare bundle to the non-singular fibres, it is the relative Poincare sheaf of the fibred product of the two families over the base B and we will call it $\mathcal{E}$. In particular, there is a relative polarization and thus we can assume that the fibration is a projective morphism. In the case the fibres are of dimension one, this equivalent to the existence of a multisection. So we will assume the existence of a multisection which means we have a smooth morphism and after an ètale base change, we get a family of abelian polarized varieties admitting a global section which outside the discriminant locus coincides with the given one.

Proposition 2.4. *The Fourier–Mukai transform*

$$\phi_{\mathcal{E}} : D^b(X|_{B-\Sigma(p)}) \to D^b\big(X^\vee|_{B-\Sigma(p)}\big)$$

with kernel $\mathcal{E}$, is an equivalence of derived categories over the smooth locus.

Proof. Let $\widetilde{B} \subset B$ be the open subset supporting the smooth fibres of p, and let $X^{sm} := X|_{B-\Sigma(p)}$ be the fibration restricted to the smooth locus $\widetilde{B} := B - \Sigma(p)$. The family has the structure of a scheme, these are the abelian schemes that can be seen as schemes in groups and have been already studied by Mukai. Due to Deligne [1], there exists a relative polarization meeting transversely any of the irreducible components of any fibre, so there is a global section $\sigma : \widetilde{B} \to X^{sm}$ obtained by associating to a point $t \in B$, the point $[\mathcal{O}_{X_t}]$ corresponding to the semistable sheaf $\mathcal{O}_{X_t}$ on the fibre X_t. The Picard functor is representable by the dual fibration $\widehat{X}^{sm} := X^\vee|_{B-\Sigma(p)}$. The dual fibration $\widehat{X}^{sm}$ is defined in such a way that the fibres correspond to the dual abelian varieties of the original fibration, that is, if $\mathcal{E}$ is the relative Poincaré sheaf, then $\forall b \in B$, $\mathcal{P}_b$ is the Poincaré bundle over $X_b \times \widehat{X}_b$. There is a natural polarization by considering the product $\pi_1^*\mathcal{O}_{X^{sm}}(\Theta) \otimes \pi_2^*\mathcal{O}_{\widehat{X}^{sm}}(\Theta)$, where $\Theta := \sigma(\widetilde{B})$ and π_1, π_2 are the projection maps of $X^{sm} \times_B \widehat{X}^{sm}$ over the first and second components. The fibres X_b and $\widehat{X}_b$ are derived equivalent, and the equivalence is given by the Poincaré bundle over the product $X_b \times \widehat{X}_b$.

Then the Fourier–Mukai transform $\phi_{\mathcal{E}} : D^b(X^{sm}) \to D^b(\widehat{X}^{sm})$ with kernel $\mathcal{E}$, defines an equivalence of the corresponding derived categories. Thus the abelian schemes X^{sm} and $\widehat{X}^{sm}$ are derived equivalent and the equivalence is given by the FMT with kernel the relative Poincaré sheaf,

$$\phi_{\mathcal{E}}(L) = \mathbf{R}\pi_{2*}\big(\pi_1^*L \otimes \mathcal{E}\big). \qquad \square$$

Definition 2.5. The dual fibration $(X^\vee/B)$ is defined as the moduli stack representing the extended Poincaré sheaf $\mathcal{E}$.

If $\mathcal{J}$ is the relative moduli functor, and $Pic^0(X/B)$, the relative Jacobian, that is, the variety $X^\vee$ representing the relative moduli functor, the relative Poincaré sheaf $\mathcal{E}$ is the family representing an element of $\mathcal{J}(Pic^0(X/B))$ such that for each variety S and each $\mathcal{F} \in \mathcal{J}(S)$ there exists a unique morphism $f : S \to X^\vee$ satisfying that $\mathcal{F} \cong f^*\mathcal{E}$. Therefore $\mathcal{E}$ induces a natural transformation $\Phi : \mathcal{J} \to Hom(-, X/B)$ giving a stack structure $((X/B), \Phi)$.

It is universal in the sense that for every other variety N and every natural transformation

$$\chi : Hom(-, N) \to Hom\big(-, X^\vee\big),$$

the following diagram commutes:

$$\begin{tikzcd}
\mathcal{J} \arrow[rr, "\Phi"] \arrow[dr] & & Hom(-, N) \arrow[dl] \\
& Hom(-, X^\vee) &
\end{tikzcd}$$

Theorem 2.6. *The extended relative Poincaré sheaf $\mathcal{E}$ to the total fibration induces a derived equivalence between X/B and $X^\vee/B$.*

Proof. Since we are assuming there is a multisection $m : B \to X$, the fibration is a smooth morphism and we can consider an ètale finite covering $\tau : B' \to B$ of the base B (that is, locally around a point $y \in B'$ and $x = \tau(y)$, τ is simply the function $\{z \in \mathbb{C}\colon\ |z| < 1\} \to \{z \in \mathbb{C} \mid |z| < 1\}$ given by $z \to z^k$, where k is the multiplicity of τ at y). Moreover, we can assume that $\tau : B' \to B$ is a Galois covering with finite Galois group G, just we observe that any normal extension of fields admits a Galois extension. If $\mathcal{B}, \mathcal{B}'$ are the fields of meromorphic functions on B and B' respectively, $\tau^* : \mathcal{B} \to \mathcal{B}'$ is a Galois field extension of degree k, with Galois group G (acting by pull-back on $\mathcal{B}'$).

$$\begin{tikzcd} X' = X \times_B B' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ B' \arrow[r] \arrow[u, bend left, "s"] & B \end{tikzcd}$$

Now the fibration we get $X' \to B'$ admits a section s that is the pull-back $(\tau \circ m)^*$ of the multisection, and therefore there is a relative Poincaré sheaf $\mathcal{E}$ as in the proof of Proposition 2.4, that restricted on smooth fibres $X'_b \times \widehat{X}'_b$, where $\widehat{X}'_b$ is the corresponding dual abelian variety, is just the Poincare bundle. We are taking as dual fibration of X'/B', the relative moduli space. If $\mathcal{J}$ is the relative moduli functor, $\mathcal{J} : \mathit{VAR} \to \mathit{SETS}$, of semistable sheaves of the fibers containing line bundles of degree 0 on smooth fibres, over the smooth locus, $\mathcal{J}$ is represented by the relative Jacobian $\mathit{Pic}^0(X'/B')$, which is the dual fibration $\widehat{X}'^{sm}/B'$.

Due to the presence of singular fibres, the corresponding coarse moduli space is not a fine moduli space, but the stack maybe a FM partner, using the relative Poincaré sheaf $\mathcal{E}$ as kernel of the FM transform, that is, the relative Poincaré sheaf $\mathcal{E}$ is the family representing an element of $\mathcal{J}(\mathit{Pic}^0(X'/B'))$ such that for each variety S and each object $\mathcal{F} \in \mathcal{J}(S)$ there exists a unique morphism $f : S \to \widehat{X}'$ satisfying that $\mathcal{F} \cong f^*\mathcal{E}$.

Thus there is an equivalence of categories

$$D^b\big(X'/B'\big) \cong D^b\big(\widehat{X}'/B'\big),$$

defined by the extended Poincaré sheaf $\mathcal{E}$ where $\widehat{\rho} : \widehat{X}' \to B'$ is the dual abelian fibration. Now the Galois group G acts on bundles on the fibres, and since they are invariant under the Galois action, there is an equivalence between the respective invariant subcategories $(D^b(X'/B'))^G \cong (D^b(\widehat{X}'/B'))^G$ by the action of the Galois group, therefore by the fundamental theorem of Galois theory, there is an equivalence between the original categories $D^b(X/B) \cong D^b(\widehat{X}/B)$. □

Corollary 2.7. *There is an equivalence* $\phi_b : D^b(X_b) \to D^b(\widehat{X}_b)$ *for every closed point* $b \in B$.

Proof. By Theorem 2.6, the integral functor $\phi_{\mathcal{E}}^{X \to \widehat{X}} : D^b(X) \to D^b(\widehat{X})$ is an equivalence of derived categories, where $\widehat{\rho} : \widehat{X} \to B$ is the dual abelian fibration. It follows from Prop. 2.15 of [4] that there is fibrewise equivalence $\phi_b : D^b(X_b) \to D^b(\widehat{X}_b)$. □

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