Optimal Control

# Stochastic maximum principle for optimal control of SPDEs ☆

## Sur le principe du maximum stochastique pour le contrôle optimal des EDP stochastiques

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A R T I C L E   I N F O

A B S T R A C T

In this Note, we give the stochastic maximum principle for optimal control of stochastic PDEs in the general case (when the control domain need not be convex and the diffusion coefficient can contain a control variable).

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R É S U M É

Dans cette Note, nous présentons un principe du maximum stochastique pour le contrôle optimal des EDP stochastiques dans le cas général (quand le domaine du contrôle n'est pas forcément convexe et que le coefficient de diffusion peut contenir la variable de contrôle).

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**Version française abrégée**

Trouver des conditions nécessaires d'optimalité pour les problèmes de contrôle optimal stochastique (généralisant dans cette manière le principe de maximum de Pontryagin au cas stochastique), est un problème qui a été résolu en grande généralité, dans le cas classique de dimension finie, dans un papier bien connu par S. Peng [5]. L'auteur permet à l'ensemble des contrôles d'être non-convexe et au coefficient de diffusion de dépendre du contrôle. En ce qui concerne les équations infinie-dimensionnelles, le cas où le domaine du contrôle est convexe et le cas où la diffusion ne dépend pas du contrôle ont été traités par [1,2]. Au contraire, dans le cas général (quand le domaine du contrôle n'est pas forcément convexe et que les coefficients de diffusion peuvent contenir la variable de contrôle) les résultats existant sont limités aux équations abstraites d'évolution qui ne sont pas satisfaites par la majorité des EDP stochastiques concrètes (par exemple le cas des opérateurs de Nemitsky sur l'espace $L^p$ n'est pas couvert, voir [6,3]). Ici nous formulons une EDP stochastique contrôlée de façon semi-abstraite et montrons comment les propriétés spécifiques de régularité sur le semigroupe correspondant à l'opérateur différentiel peuvent être utilisées pour traiter le cas des coefficients de type Nemitsky. Le point clé est la caractérisation explicite du second processus dual en terme d'une fonctionnelle quadratique appropriée, voir la Définition 4.1.

Les hypothèses sont données à la Section 2.

**Théorème.** *Soit $(\bar{X}_t, \bar{u}_t)$ un couple optimal et soient $p, q_1, \ldots, q_m$ définis comme dans la Proposition* 4.1 *et $P$ défini comme dans la Définition* 4.1. *Alors $\mathbb{P}$-p.s. p.p. $t \in [0, T]$ et pour tout $v \in U$ :*

$$\mathcal{H}(v, \bar{X}_t, p_t, q_{1t}, \ldots, q_{mt}) - \mathcal{H}(\bar{u}_t, \bar{X}_t, p_t, q_{1t}, \ldots, q_{mt})$$
$$+ \frac{1}{2}\langle P_t[\sigma_j(\cdot, \bar{X}_t(\cdot), v) - \sigma_j(\cdot, \bar{X}_t(\cdot), \bar{u}_t)], \sigma_j(\cdot, \bar{X}_t(\cdot), v) - \sigma_j(\cdot, \bar{X}_t(\cdot), \bar{u}_t)\rangle \geqslant 0,$$

*où*

$$\mathcal{H}(u, X, p, q_1, \ldots, q_m) = \int_{\mathcal{O}} \left[l\big(x, X(x), u\big) + b\big(x, X(x), u\big)p(x) + \sigma_j\big(x, X(x), u\big)q_j(x)\right] dx.$$

## 1. Introduction

The problem of finding necessary optimality conditions for stochastic optimal control problems (generalizing in this way the Pontryagin maximum principle to the stochastic case) has been solved in great generality, in the classical finite-dimensional case, in the well known paper by S. Peng [5]. The author allows the set of control actions to be non-convex and the diffusion coefficient to depend on the control; consequently he is led to introducing the equations for the second variation process and for its dual. As far as infinite-dimensional equations are concerned the cases in which the control domain is convex or diffusion does not depend on the control have been treated in [1,2]. On the contrary in the general case (when the control domain need not be convex and the diffusion coefficient can contain a control variable) existing results are limited to abstract evolution equations under assumptions that are not satisfied by the large majority of concrete stochastic PDEs (for instance the case of Nemitsky operators on $L^p$ spaces is not covered, see [6,3]). Here we formulate a controlled parabolic stochastic PDE in a semi-abstract way and show how the specific regularity properties on the semi-group corresponding to the differential operator can be used to treat the case of Nemitsky-type coefficients. The key point is the explicit characterization of the second dual process in term of a suitable quadratic functional, see Definition 4.1.

## 2. Formulation of the optimal control problem

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open set with regular boundary. We consider the following controlled SPDE formulated in a partially abstract way in the state space $H = L^2(\mathcal{O})$ (norm $|\cdot|$, scalar product $\langle\cdot,\cdot\rangle$):

$$\begin{cases} dX_t(x) = AX_t(x)\,dt + b\big(x, X_t(x), u_t\big)\,dt + \sum_{j=1}^{m} \sigma_j\big(x, X_t(x), u_t\big)\,d\beta_t^j, & t \in [0, T],\ x \in \mathcal{O}, \\ X_0(x) = x_0(x), \end{cases} \tag{1}$$

and the following cost functional:

$$J(u) = \mathbb{E}\int_0^T \int_{\mathcal{O}} l\big(x, X_t(x), u_t\big)\,dx\,dt + \mathbb{E}\int_{\mathcal{O}} h\big(x, X_T(x)\big)\,dx.$$

We work in the following setting:

**Hypothesis 2.1.**

1. $A$ is the realization of a partial differential operator with appropriate boundary conditions. We assume that $A$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$, $t \geqslant 0$, in $H$. Moreover, for every $p \in [2, \infty)$ and $t \in [0, T]$, $e^{tA}(L^p(\mathcal{O})) \subset L^p(\mathcal{O})$ with $\|e^{tA}f\|_{L^p(\mathcal{O})} \leqslant C_{p,T}\|f\|_{L^p(\mathcal{O})}$ for some constants $C_{p,T}$ independent of $t$ and $f$. Finally the restriction of $e^{tA}$, $t \geqslant 0$, to $L^4(\mathcal{O})$ is an analytic semigroup with domain of the infinitesimal generator compactly embedded in $L^4(\mathcal{O})$.
2. $(\beta_t^1, \ldots, \beta_t^m)$, $t \geqslant 0$ is a standard $m$-dimensional Wiener process on a complete probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and we denote by $(\mathcal{F}_t)_{t \geqslant 0}$ its natural (completed) filtration. All stochastic processes will be progressively measurable with respect to $(\mathcal{F}_t)_{t \geqslant 0}$.
3. $b, \sigma_j$ $(j = 1, \ldots, m)$, $l : \mathcal{O} \times \mathbb{R} \times U \to \mathbb{R}$ and $h : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ are measurable functions. We assume that they are continuous with respect to the third variable (the control variable), of class $C^2$ with respect to the second (the state variable), and bounded together with their first and second derivative with respect to the second variable.
4. The set of admissible control actions is a separable metric space $U$ and an admissible control $u$ is a (progressive) process with values in $U$.

Under the above conditions, for every control $u$ there exists a unique mild solution, i.e. a continuous process in $H$ such that, $\mathbb{P}$-a.s.

$$X_t = e^{tA}x_0 + \int_0^t e^{(t-s)A}b\big(\cdot, X_s(\cdot), u_s\big)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\sigma_j\big(\cdot, X_s(\cdot), u_s\big)\,\mathrm{d}\beta_s^j, \quad t \in [0, T].$$

**Example 2.1.** Let $\mathcal{O} \subset \mathbb{R}^n$ be an open bounded domain with uniformly $C^2$ boundary $\partial\mathcal{O}$. Consider the following second order differential operator:

$$(Af)(x) = \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n c_i(x)\frac{\partial f}{\partial x_i}(x) + c_0(x)f(x), \quad x \in \mathcal{O},$$

with either Dirichlet boundary conditions $f(x) = 0$, $x \in \partial\mathcal{O}$ or oblique boundary conditions

$$\sum_{i=1}^n \gamma_i(x)\frac{\partial f}{\partial x_i}(x) + \gamma_0(x)f(x) = 0, \quad x \in \partial\mathcal{O}.$$

Assume that $a_{i,j}$, $c_i$, $c_0: \mathbb{R}^n \to \mathbb{R}$ are uniformly continuous and bounded and (if we choose oblique boundary conditions) that $\gamma_i$, $\gamma_0: \mathbb{R}^n \to \mathbb{R}$ and their first order derivatives are uniformly continuous and bounded.

It is shown in [4, Section 3.1.1] that under the usual uniform ellipticity condition:

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \text{for some } \lambda > 0 \text{ and all } x \in \bar{\mathcal{O}},\ \xi \in \mathbb{R}^n$$

and (if we choose oblique boundary conditions) a uniform nontangentiality condition

$$\inf_{x \in \partial\mathcal{O}}\left|\sum_{i=1}^n \gamma_i(x)\nu_i(x)\right| > 0, \quad \nu \text{ being the exterior unit normal vector to } \partial\mathcal{O},$$

$A$ generates an analytic semigroup in all $L^p(\mathcal{O})$, $1 < p < \infty$, with domain compactly embedded in $L^p(\mathcal{O})$ and thus fulfills the assumptions in Hypothesis 2.1.1.

Further examples could be given by elliptic operators of order $2m$ with suitable regular coefficients (see [4, Section 3.2] and references within).

## 3. Expansions of the solution and of the cost

We assume that an optimal control $\bar{u}$ exists and denote by $\bar{X}$ the corresponding optimal state. We introduce the spike variation: we fix an arbitrary interval $[\bar{t}, \bar{t}+\epsilon] \subset (0, T)$ and an arbitrary $U$-valued, $\mathcal{F}_{\bar{t}}$-measurable random variable $v$, and define the following perturbation of $\bar{u}$: $u_t^\epsilon = vI_{[\bar{t},\bar{t}+\epsilon]}(t) + \bar{u}_tI_{[\bar{t},\bar{t}+\epsilon]^c}(t)$ and denote by $X^\epsilon$ the solution of the state equation (1) with control $u = u^\epsilon$.

We introduce two linear equations corresponding to first and second expansion of $X^\epsilon$ with respect to $\epsilon$ (both equations are understood in the mild sense). In the following, derivatives with respect to the state variable will be denoted $b'$, $b''$, $\sigma'$, $\sigma''$ and

$$\delta b_t(x) = b\big(x, \bar{X}_t(x), u_t^\epsilon\big) - b\big(x, \bar{X}_t(x), \bar{u}_t\big), \qquad \delta\sigma_{jt}(x) = \sigma_j\big(x, \bar{X}_t(x), u_t^\epsilon\big) - \sigma_j\big(x, \bar{X}_t(x), \bar{u}_t\big),$$

$$\delta b_t'(x) = b'\big(x, \bar{X}_t(x), u_t^\epsilon\big) - b'\big(x, \bar{X}_t(x), \bar{u}_t\big), \qquad \delta\sigma_{jt}'(x) = \sigma_j'\big(x, \bar{X}_t(x), u_t^\epsilon\big) - \sigma_j'\big(x, \bar{X}_t(x), \bar{u}_t\big).$$

Consider

$$\begin{cases} \mathrm{d}Y_t^\epsilon(x) = \big[AY_t^\epsilon(x) + b'\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Y_t^\epsilon(x)\big]\mathrm{d}t + \sigma_j'\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Y_t^\epsilon(x)\,\mathrm{d}\beta_t^j + \delta b_t(x)\,\mathrm{d}t + \delta\sigma_{jt}(x)\,\mathrm{d}\beta_t^j, \\ Y_0^\epsilon(x) = 0, \end{cases} \quad (2)$$

$$\begin{cases} \mathrm{d}Z_t^\epsilon(x) = \big[AZ_t^\epsilon(x) + b'\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Z_t^\epsilon(x)\big]\mathrm{d}t + \sigma_j'\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Z_t^\epsilon(x)\,\mathrm{d}\beta_t^j \\ \qquad\quad + \left[\frac{1}{2}b''\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Y_t^\epsilon(x)^2 + \delta b_t'(x) \cdot Y_t^\epsilon(x)\right]\mathrm{d}t \\ \qquad\quad + \left[\frac{1}{2}\sigma_j''\big(x, \bar{X}_t(x), \bar{u}_t\big) \cdot Y_t^\epsilon(x)^2 + \delta\sigma_{jt}'(x) \cdot Y_t^\epsilon(x)\right]\mathrm{d}\beta_t^j, \\ Z_0^\epsilon(x) = 0. \end{cases} \quad (3)$$

We notice that to formulate the second equation in $H$ we need to show that the first admits solutions in $L^4(\mathcal{O})$.

The following proposition states existence and uniqueness of the solution to the above equation in all spaces $L^p(\mathcal{O})$ together with the estimate of their dependence with respect to $\epsilon$. The proof is technical but based on standard estimates and we omit it.

**Proposition 3.1.** *Eqs.* (2) *and* (3) *admit a unique continuous mild solution. Moreover for all $p \geqslant 2$*

$$\sup_{t \in [0,T]} \left(\sqrt{\epsilon}^{-1}\left(\mathbb{E}\|Y_t^\epsilon\|_{L^p(\mathcal{O})}^p\right)^{1/p} + \epsilon^{-1}\left(\mathbb{E}\|Z_t^\epsilon\|_{L^p(\mathcal{O})}^p\right)^{1/p}\right) \leqslant C_p, \qquad \sup_{t \in [0,T]}\left(\mathbb{E}\|X_t^\epsilon - \bar{X}_t - Y_t^\epsilon - Z_t^\epsilon\|_H^2\right)^{1/2} = o(\epsilon).$$

As far as the cost is concerned we set $\delta l_t(x) = l(x, \bar{X}_t(x), u_t^\epsilon) - l(x, \bar{X}_t(x), \bar{u}_t)$ and prove that

**Proposition 3.2.**

$$J(u^\epsilon) - J(\bar{u}) = \mathbb{E}\int_0^T \int_{\mathcal{O}} \delta l_t(x)\, dx\, dt + \Delta_1^\epsilon + \Delta_2^\epsilon + o(\epsilon),$$

*where*

$$\Delta_1^\epsilon = \mathbb{E}\int_0^T \int_{\mathcal{O}} l'\left(x, \bar{X}_t(x), \bar{u}_t\right)\left(Y_t^\epsilon(x) + Z_t^\epsilon(x)\right) dx\, dt + \mathbb{E}\int_{\mathcal{O}} h'\left(x, \bar{X}_T(x)\right)\left(Y_T^\epsilon(x) + Z_T^\epsilon(x)\right) dx,$$

$$\Delta_2^\epsilon = \frac{1}{2}\mathbb{E}\int_0^T \int_{\mathcal{O}} l''\left(x, \bar{X}_t(x), \bar{u}_t\right)Y_t^\epsilon(x)^2\, dx\, dt + \frac{1}{2}\mathbb{E}\int_{\mathcal{O}} h''\left(x, \bar{X}_T(x)\right)Y_T^\epsilon(x)^2\, dx.$$

## 4. The first and second adjoint processes

The following proposition is special case of a result in [2]:

**Proposition 4.1.** *Let $A^*$ be the $L^2(\mathcal{O})$-adjoint operator of $A$. There exists a unique $m+1$-tuple of $L^2(\mathcal{O})$-valued processes $(p, q_j)$, with $p$ continuous and $\mathbb{E}\sup_{t \in [0,T]}|p_t|^2 + \mathbb{E}\int_0^T |q_{jt}|^2\, dt < \infty$, that verify (in a mild sense) the backward stochastic differential equation:*

$$\begin{cases} -dp_t(x) = -q_{jt}(x)\, d\beta_t^j + \left[A^* p_t(x) + b'\left(x, \bar{X}_t(x), \bar{u}_t\right) \cdot p_t(x) + \sigma_j'\left(x, \bar{X}_t(x), \bar{u}_t\right) \cdot q_{jt}(x) + l'\left(x, \bar{X}_t(x), \bar{u}_t\right)\right] dt, \\ p_T(x) = h'\left(x, \bar{X}_T(x)\right). \end{cases}$$

The following proposition formally follows from Proposition 3.2 computing the Itô differentials $d\int_{\mathcal{O}} Y_t^\epsilon(x)p_t(x)\, dx$ and $d\int_{\mathcal{O}} Z_t^\epsilon(x)p_t(x)\, dx$, while the formal proof goes through Yosida approximations of $A$.

**Proposition 4.2.** *We have*

$$J(u^\epsilon) - J(u) = \mathbb{E}\int_0^T \int_{\mathcal{O}} \left[\delta l_t(x) + \delta b_t(x)p_t(x) + \delta\sigma_{jt}(x)q_{jt}(x)\right] dx\, dt + \frac{1}{2}\Delta_3^\epsilon + o(\epsilon), \tag{4}$$

*where*

$$\Delta_3^\epsilon = \mathbb{E}\int_0^T \int_{\mathcal{O}} \bar{H}_t(x)Y_t^\epsilon(x)^2\, dx\, dt + \mathbb{E}\int_{\mathcal{O}} \bar{h}(x)Y_T^\epsilon(x)^2\, dx, \tag{5}$$

*with*

$$\bar{H}_t(x) = l''\left(x, \bar{X}_t(x), \bar{u}_t\right) + b''\left(x, \bar{X}_t(x), \bar{u}_t\right)p_t(x) + \sigma_j''\left(x, \bar{X}_t(x), \bar{u}_t\right)q_{jt}(x), \qquad \bar{h}(x) = h''\left(x, \bar{X}_T(x)\right).$$

*We notice that the multiplication by $\bar{H}_t(\cdot)$ is not a bounded operator in $H$.*

**Definition 4.1.** For fixed $t \in [0, T]$ and $f \in L^4(\mathcal{O})$, we consider the equation (understood as usual in mild form)

$$\begin{cases} dY_s^{t,f}(x) = AY_s^{t,f}(x)\, ds + b'\left(x, \bar{X}_s(x), \bar{u}_s\right)Y_s^{t,f}(x)\, ds + \sigma_j'\left(x, \bar{X}_s(x), \bar{u}_s\right)Y_s^{t,f}(x)\, dW_s^j, \quad s \in [t, T], \\ Y_t^{t,f}(x) = f(x). \end{cases} \tag{6}$$

We denote $\mathcal{L}$ the space of bounded linear operators $L^4(\mathcal{O}) \to L^4(\mathcal{O})^* = L^{4/3}(\mathcal{O})$ and define a progressive process $(P_t)_{t \in [0,T]}$ with values in $\mathcal{L}$ setting for $t \in [0, T]$, $f, g \in L^4(\mathcal{O})$,

$$\langle P_t f, g \rangle = \mathbb{E}^{\mathcal{F}_t} \int_t^T \int_{\mathcal{O}} \bar{H}_s(x) Y_s^{t,f}(x) Y_s^{t,g}(x) \, dx \, ds + \mathbb{E}^{\mathcal{F}_t} \int_{\mathcal{O}} \bar{h}(x) Y_T^{t,f}(x) Y_T^{t,g}(x) \, dx, \quad \mathbb{P}\text{-a.s.}$$

(by abuse of language by $\langle \cdot, \cdot \rangle$ we also denote the duality between $L^4(\mathcal{O})$ and $L^{4/3}(\mathcal{O})$).

Exploiting the analyticity of the semigroup generated by $A$ on $L^4(\mathcal{O})$ we prove the following proposition that is the key point for our final argument.

**Proposition 4.3.** *We have* $\sup_{t \in [0,T]} \mathbb{E} \|P_t\|_{\mathcal{L}}^2 < \infty$. *Moreover* $\mathbb{E}|\langle (P_{t+\epsilon} - P_t) f, g \rangle| \to 0$, *as* $\epsilon \to 0$, $\forall f, g \in L^4(\mathcal{O})$. *Finally for every* $\eta \in (0, 1/4)$ *there exists a constant* $C_\eta$ *such that*

$$\left| \langle P_t(-A)^\eta f, (-A)^\eta g \rangle \right| \leqslant C_\eta \|f\|_4 \|g\|_4 (T-t)^{-2\eta} \left[ \left( \int_t^T \mathbb{E}^{\mathcal{F}_t} |\bar{H}_s|^2 \, ds \right)^{1/2} + \left( \mathbb{E}^{\mathcal{F}_t} |\bar{h}|^2 \right)^{1/2} \right], \quad \mathbb{P}\text{-a.s.} \tag{7}$$

*where* $D(-A)^\eta$ *is the domain of the fractional power of $A$ in* $L^4(\mathcal{O})$ *and by* $\| \cdot \|_4$ *we denote the norm in* $L^4(\mathcal{O})$.

## 5. The maximum principle

For $u \in U$ and $X, p, q_1, \ldots, q_m \in L^2(\mathcal{O})$ denote

$$\mathcal{H}(u, X, p, q_1, \ldots, q_m) = \int_{\mathcal{O}} \left[ l(x, X(x), u) + b(x, X(x), u) p(x) + \sigma_j(x, X(x), u) q_j(x) \right] dx.$$

**Theorem 5.1.** *Let* $(\bar{X}_t, \bar{u}_t)$ *be an optimal pair and let* $p, q_1, \ldots, q_m$ *be defined as in Proposition* 4.1 *and $P$ be defined as in Definition* 4.1. *Then the following inequality holds* $\mathbb{P}$-a.s. *for a.e.* $t \in [0, T]$ *and for every* $v \in U$:

$$\mathcal{H}(v, \bar{X}_t, p_t, q_{1t}, \ldots, q_{mt}) - \mathcal{H}(\bar{u}_t, \bar{X}_t, p_t, q_{1t}, \ldots, q_{mt})$$
$$+ \frac{1}{2} \langle P_t [\sigma_j(\cdot, \bar{X}_t(\cdot), v) - \sigma_j(\cdot, \bar{X}_t(\cdot), \bar{u}_t)], \sigma_j(\cdot, \bar{X}_t(\cdot), v) - \sigma_j(\cdot, \bar{X}_t(\cdot), \bar{u}_t) \rangle \geqslant 0.$$

**Proof.** By the Markov property of the solutions to Eq. (6) and Proposition 3.1 we get:

$$\mathbb{E} \int_0^T \langle \bar{H}_s Y_s^\epsilon, Y_s^\epsilon \rangle \, ds + \mathbb{E} \langle \bar{h} Y_T^\epsilon, Y_T^\epsilon \rangle = \mathbb{E} \int_{t_0 + \epsilon}^T \langle \bar{H}_s Y_s^{t_0 + \epsilon, Y_{t_0 + \epsilon}^\epsilon}, Y_s^{t_0 + \epsilon, Y_{t_0 + \epsilon}^\epsilon} \rangle \, ds + \mathbb{E} \langle \bar{h} Y_T^{t_0 + \epsilon, Y_{t_0 + \epsilon}^\epsilon}, Y_T^{t_0 + \epsilon, Y_{t_0 + \epsilon}^\epsilon} \rangle + o(\epsilon)$$

$$= \mathbb{E} \langle P_{t_0 + \epsilon} Y_{t_0 + \epsilon}^\epsilon, Y_{t_0 + \epsilon}^\epsilon \rangle + o(\epsilon). \tag{8}$$

We wish to replace $P_{t_0 + \epsilon}$ by $P_{t_0}$ in the above that is we claim that $\mathbb{E} \langle (P_{t_0 + \epsilon} - P_{t_0}) Y_{t_0 + \epsilon}^\epsilon, Y_{t_0 + \epsilon}^\epsilon \rangle = o(\epsilon)$, or equivalently that $\mathbb{E} \langle (P_{t_0 + \epsilon} - P_{t_0}) \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon \rangle \to 0$.

To prove the above claim we need a compactness argument. A similar argument will allow us to approximate $P$ by suitable finite-dimensional projections at the end of this proof.

By the Markov inequality and Proposition 3.1 if we set $K_\delta = \{ f \in L^4 : f \in D(-A)^\eta, \|f\|_{D(-A)^\eta} \leqslant C_0 \delta^{-1/4} \}$, for a suitable constant $C_0$, and denote by $\Omega_{\delta, \epsilon}$ the event $\{ \epsilon^{-1/2}(-A)^{-\eta} Y_{t_0 + \epsilon}^\epsilon \in K_\delta \}$ we get $\mathbb{P}(\Omega_{\delta, \epsilon}^c) \leqslant \delta$.

Moreover

$$\mathbb{E} \langle (P_{t_0 + \epsilon} - P_{t_0}) \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon \rangle$$
$$= \mathbb{E} \left[ \langle (P_{t_0 + \epsilon} - P_{t_0}) \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon \rangle 1_{\Omega_{\delta, \epsilon}^c} \right] + \mathbb{E} \left[ \langle (P_{t_0 + \epsilon} - P_{t_0}) \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon \rangle 1_{\Omega_{\delta, \epsilon}} \right]$$
$$=: A_1^\epsilon + A_2^\epsilon.$$

By the Hölder inequality

$$\left| A_1^\epsilon \right| \leqslant \left( \mathbb{E} \|P_{t_0 + \epsilon} - P_{t_0}\|_{\mathcal{L}}^2 \right)^{1/2} \left( \mathbb{E} \|\epsilon^{-1/2} Y_{t_0 + \epsilon}^\epsilon\|_4^8 \right)^{1/4} \mathbb{P}(\Omega_{\delta, \epsilon}^c)^{1/4},$$

and from the estimates in Proposition 4.3 we conclude that $|A_1^\epsilon| \leqslant c \mathbb{P}(\Omega_{\delta, \epsilon}^c)^{1/4} = O(\delta^{1/4})$.

On the other hand, recalling the definition of $\Omega_{\delta, \epsilon}$, $|A_2^\epsilon| \leqslant \mathbb{E} \sup_{f \in K_\delta} |\langle (P_{t_0 + \epsilon} - P_{t_0}) (-A)^\eta f, (-A)^\eta f \rangle 1_{\Omega_{\delta, \epsilon}}|$.

Since $K_\delta$ is compact in $L^4$, it can be covered by a finite number $N_\delta$ of open balls with radius $\delta$ and centers denoted $f_i^\delta$, $i = 1, \ldots, N_\delta$. Since $D(-A)^\eta$ is dense in $L^4$, we can assume that $f_i^\delta \in D(-A)^\eta$. Given $f \in K_\delta$, let $i$ be such that $\|f - f_i^\delta\|_4 < \delta$; then writing

$$\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f, (-A)^\eta f \rangle = \langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f_i^\delta, (-A)^\eta f_i^\delta \rangle$$
$$- \langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta (f - f_i^\delta), (-A)^\eta (f - f_i^\delta) \rangle$$
$$+ 2\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f, (-A)^\eta (f - f_i^\delta) \rangle$$

and taking expectation, it follows from (7) that

$$\left| A_2^\epsilon \right| \leqslant \sum_{i=1}^{N_\delta} \mathbb{E} \left| \langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f_i^\delta, (-A)^\eta f_i^\delta \rangle \right| + c(T - t_0 - \epsilon)^{-2\eta} \left[ \delta^2 + \delta^{3/4} \right],$$

and by the second statement in Proposition 4.3 we conclude that

$$\limsup_{\epsilon \downarrow 0} \left| A_2^\epsilon \right| \leqslant c(T - t_0)^{-2\eta} \left[ \delta^2 + \delta^{3/4} \right].$$

Letting $\delta \to 0$ we obtain $|A_1^\epsilon| + |A_2^\epsilon| \to 0$ and the proof that $\mathbb{E}\langle (P_{t_0+\epsilon} - P_{t_0})\epsilon^{-1/2} Y_{t_0+\epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0+\epsilon}^\epsilon \rangle \to 0$ is finished.

We finally show that

$$\mathbb{E}\langle P_{t_0} Y_{t_0+\epsilon}^\epsilon, Y_{t_0+\epsilon}^\epsilon \rangle = \mathbb{E} \int_{t_0}^{t_0+\epsilon} \langle P_s \delta^\epsilon \sigma_j(s, \cdot), \delta^\epsilon \sigma_j(s, \cdot) \rangle \, ds + o(\epsilon).$$

If we treat $A$ and $P_{t_0}$ as bounded operators in $H$ we get, by Itô rule:

$$\mathbb{E}\langle P_{t_0} Y_{t_0+\epsilon}^\epsilon, Y_{t_0+\epsilon}^\epsilon \rangle = 2\mathbb{E} \int_{t_0}^{t_0+\epsilon} \langle P_{t_0} Y_s^\epsilon, (A + b'(\bar{X}_t, \bar{u}_t)) Y_s^\epsilon \rangle \, ds + \mathbb{E} \int_{t_0}^{t_0+\epsilon} \langle P_{t_0} \delta^\epsilon \sigma_j(s, \cdot), \delta^\epsilon \sigma_j(s, \cdot) \rangle \, ds$$

and the claim follows recalling that $\mathbb{E}|Y_t^\epsilon|^2 = O(\epsilon)$ and the "continuity" of $P$ stated in Proposition 4.3.

The general case is more technical and requires a double approximation: $A$ by its Yosida Approximations and $P$ by finite-dimensional projections $P_t^N(\omega)f := \sum_{i,j=1}^N \langle P_t(\omega)e_i, e_j \rangle \langle e_i, f \rangle_2 e_j$, $f \in L^4$, where $(e_i)_{i \geqslant 1}$ is an orthonormal basis in $L^2$ which is also a Schauder basis of $L^4$.

The conclusion of the proof of the maximum principle is now standard (see, e.g. [2,6] or [5]). We just have to write $J(u^\epsilon) - J(u)$ using (4), (5) and (8), to recall that $0 \leqslant \epsilon^{-1}(J(u^\epsilon) - J(u))$ and to let $\epsilon \to 0$. $\quad \square$

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