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Partial Differential Equations

Comments on two Notes by L. Ma and X. Xu

Commentaires sur deux Notes de L. Ma et X. Xu

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ABSTRACT

In this Note I discuss some assertions made by L. Ma and X. Xu (2009) [6] and L. Ma (2010) [5], which need to be corrected and supplemented with additional references.

RÉSUMÉ

Dans cette Note j'apporte des corrections et des références supplémentaires à des assertions de L. Ma et X. Xu (2009) [6] et L. Ma (2010) [5].

1. Main results

Li Ma and Xingwang Xu [6,5], considered positive smooth solutions u of the equation

$$\Delta u = u^q - u^{-q-2} \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $q > 0$. The following assertion can be found in [5]:

$$\text{the only solution of (1) is } u \equiv 1. \tag{A_q}$$

It turns out that assertion (A_q) is not quite correct. More precisely, we have

Claim 1. *If $q > 1$, assertion (A_q) holds and follows easily from the Keller–Osserman theory* [3,7].

Claim 2. *When $0 < q \leqslant 1$, assertion (A_q) fails: Eq. (1) admits many solutions.*

First we observe that

$$\text{any solution of (1) with } q > 0 \text{ satisfies } u \geqslant 1 \text{ in } \mathbb{R}^N. \tag{2}$$

Proof of (2). The argument is standard. Set $f(t) = t^q - t^{-q-2}$, $t > 0$. Fix any $x_0 \in \mathbb{R}^N$ and consider the function

$$u_\varepsilon(x) = u(x) + \varepsilon|x - x_0|^2, \quad \varepsilon > 0,\ x \in \mathbb{R}^N.$$

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Since $u_\varepsilon(x) \to \infty$ as $|x| \to +\infty$, $\mathrm{Min}_{\mathbb{R}^N} u_\varepsilon$ is achieved at some x_1. We have

$$0 \leqslant \Delta u_\varepsilon(x_1) = \Delta u(x_1) + 2\varepsilon N = f\big(u(x_1)\big) + 2\varepsilon N. \tag{3}$$

On the other hand

$$u(x_1) + \varepsilon|x_1 - x_0|^2 = u_\varepsilon(x_1) \leqslant u_\varepsilon(x_0) = u(x_0),$$

and thus $u(x_1) \leqslant u(x_0)$. Since f is increasing we deduce that

$$f\big(u(x_1)\big) \leqslant f\big(u(x_0)\big). \tag{4}$$

Combining (3) and (4) and letting $\varepsilon \to 0$ yields $f(u(x_0)) \geqslant 0$, which implies (2). □

Proof of Claim 1. Set $v = u - 1 \geqslant 0$. By (1) we have

$$\Delta v = g(v) \quad \text{in } \mathbb{R}^N, \tag{5}$$

where $g(v) = (v+1)^q - (v+1)^{-q-2}$ satisfies $g(v) \geqslant v^q$ for all $v \geqslant 0$, since $q \geqslant 1$. We may then apply the Keller–Osserman theory (see [3,7], the earlier references therein, and also [4,1]), which holds for $q > 1$, to conclude that $v \equiv 0$, i.e. $u \equiv 1$. □

We now turn to Claim 2, which is an easy consequence of the following:

Lemma 3. *Assume* $0 < q \leqslant 1$. *Given any* $\alpha > 1$, *there exists a unique globally defined solution* φ *of the ODE*

$$\begin{cases} \varphi'' = f(\varphi) & \text{on } \mathbb{R},\ \varphi > 1 \text{ on } \mathbb{R}, \\ \varphi(0) = \alpha, & \varphi'(0) = 0. \end{cases} \tag{6}$$

Moreover $\varphi(-t) = \varphi(t)\ \forall t \in \mathbb{R}$ *and* $\varphi(t) \to +\infty$ *as* $t \to +\infty$.

Proof. Set $\tilde f(\xi) = f(\xi)$ if $\xi \geqslant 1$ and $\tilde f(\xi) = 0$ if $\xi \in \mathbb{R}$, $\xi \leqslant 1$. Note that $\tilde f$ is Lipschitz on $\mathbb{R}$ because $q \leqslant 1$. Hence the initial value problem, $\varphi'' = \tilde f(\varphi)$ on $\mathbb{R}$, $\varphi(0) = \alpha$, $\varphi'(0) = 0$ admits a unique globally defined solution. Since $\tilde f \geqslant 0$ on $\mathbb{R}$ we deduce that φ is convex and that $\varphi(t) \geqslant \alpha\ \forall t \in \mathbb{R}$. Therefore φ solves (6) and satisfies the required properties. □

Remark 1. The error in [5] comes from the fact that the author invokes Proposition 2 of [6] to assert that solutions of (1) are uniformly bounded. Without providing detailed computations, they use an argument in the spirit of Keller–Osserman which is valid only for $q > 1$. Lemma 1 above shows that Proposition 2 of [6] is wrong when $0 < q \leqslant 1$.

Remark 2. Using the same argument as in Lemma 1 one can obtain a globally defined solution $\psi(r)$ of the ODE

$$\begin{cases} \psi'' + \frac{N-1}{r}\psi' = f(\psi) & \text{on } (0,+\infty),\ \psi > 1 \text{ on } (0,+\infty), \\ \psi(0) = \alpha, & \psi'(0) = 0, \end{cases} \tag{7}$$

which satisfies in addition $\psi(r) \to +\infty$ as $r \to +\infty$. Then $u(x) = \psi(|x|)$ is a solution of (1) such that $u(x) \to +\infty$ as $|x| \to \infty$. Here is an interesting

Open problem. Is it true that all solutions u of (1) such that $u(x) \to +\infty$ as $|x| \to \infty$ are radial about some point in $\mathbb{R}^N$ (and therefore coincide with the solutions constructed above)?

Added in proof. Louis Dupaigne has informed me that he has constructed a counterexample to the above open problem when $q = 1$, i.e., there exist non-radial solutions of Eq. (1) which blow up at infinity. The problem remains open when $0 < q < 1$.

Remark 3. In [5], L. Ma also considers solutions of the Ginzburg–Landau equation

$$-\Delta u = u\big(1 - |u|^2\big) \quad \text{in } \mathbb{R}^N, \tag{8}$$

where $u : \mathbb{R}^N \to \mathbb{R}^k$, and he proves that u satisfies $|u| \leqslant 1$ in $\mathbb{R}^N$. This fact was originally established in 1994 by M. Hervé and R.M. Hervé [2] for $N = 2$ and $k = 2$. Shortly afterwards I noticed (unpublished) that the same conclusion holds for any N and any k as an immediate consequence of the Keller–Osserman theory via Kato's inequality (as in [1]). Indeed $\varphi = (|u|^2 - 1)^+$ satisfies, by Kato's inequality,

$$\begin{aligned}\Delta\varphi &\geqslant \big(\Delta|u|^2\big)\operatorname{sign}^+\big(|u|^2-1\big) = 2\big(u\Delta u + |\nabla u|^2\big)\operatorname{sign}^+\big(|u|^2-1\big)\\ &\geqslant 2|u|^2\big(|u|^2-1\big)\operatorname{sign}^+\big(|u|^2-1\big) \quad \text{by (8)}\\ &= 2\varphi(\varphi+1) \geqslant 2\varphi^2.\end{aligned}$$

Applying once more Keller–Osserman yields $\varphi \equiv 0$.

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