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Very nilpotent basis and n-tuples in Borel subalgebras

Bases fortement nilpotentes et n-uplets des algèbres de Borel

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A B S T R A C T

A (vector space) basis B of a Lie algebra is said to be very nilpotent if all the iterated brackets of elements of B are nilpotent. In this Note, we prove a refinement of Engel's Theorem. We show that a Lie algebra has a very nilpotent basis if and only if it is a nilpotent Lie algebra. When $\mathfrak{g}$ is a semisimple Lie algebra, this allows us to define an ideal of $S((\mathfrak{g}^n)^*)^G$ whose associated algebraic set in $\mathfrak{g}^n$ is the set of n-tuples lying in a same Borel subalgebra.

R É S U M É

Une base (d'espace vectoriel) B d'une algèbre de Lie est dite fortement nilpotente si tous les crochets itérés des éléments de B sont nilpotents. Dans cette Note, on démontre une version améliorée du théorème d'Engel. On montre qu'une algèbre de Lie admet une base fortement nilpotente si et seulement si c'est une algèbre nilpotente. Lorsque $\mathfrak{g}$ est une algèbre de Lie semi-simple, ceci nous permet de définir un idéal de $S((\mathfrak{g}^n)^*)^G$ dont l'ensemble algèbrique associé dans $\mathfrak{g}^n$ est l'ensemble des n-uplets vivants dans une même sous-algèbre de Borel.

1. Introduction and notation

Let $\mathfrak{g}$ be a Lie algebra defined over an algebraically closed field $\Bbbk$ of characteristic 0. The adjoint action of $z \in \mathfrak{g} : x \mapsto [z, x]$ is denoted by $\mathrm{ad}_z \in \mathfrak{gl}(\mathfrak{g})$. If $h \in \mathfrak{g}$, we denote the centralizer of h in $\mathfrak{g}$ by $\mathfrak{g}^h$. When V is a vector space over $\Bbbk$, and $A \subset V$, $\langle A \rangle$ stands for the linear subspace spanned by A. The symmetric algebra on V is denoted by $S(V)$. Any subset $J \subset S(V^*)$ defines an algebraic subset $\mathcal{V}(J) := \{x \in V \mid f(x) = 0, \ \forall f \in J\}$.

For $n \in \mathbb{N}^*$, let $\mathcal{I}_n$ be the set of morphisms of varieties $\mathfrak{g}^n \to \mathfrak{g}$ defined by induction as follows:

- For $i \in [\![1, n]\!]$, $((y_1, \dots, y_n) \mapsto y_i) \in \mathcal{I}_n$.
- If $f, g \in \mathcal{I}_n$, then $[f, g] := ((y_1, \dots, y_n) \mapsto [f(y_1, \dots, y_n), g(y_1, \dots, y_n)]) \in \mathcal{I}_n$.

In particular, $\mathcal{I}(y_1, \dots, y_n) := \{f(y_1, \dots, y_n) \mid f \in \mathcal{I}_n\}$ is the set of iterated brackets in $y_1, \dots, y_n$. One defines the depth map on $\mathcal{I}_n$ by induction:

$$dep\big((y_1, \dots, y_n) \mapsto y_i\big) = 1, \qquad dep[f, g] = max\{dep\, f, dep\, g\} + 1.$$

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We say that $(y_1,\dots,y_n)$ is a *very nilpotent* basis of $\mathfrak{g}$ if the following two conditions hold:

- $(y_1,\dots,y_n)$ is a basis of the vector space $\mathfrak{g}$,
- ad_z is nilpotent in $\mathfrak{gl}(\mathfrak{g})$ for any $z\in\mathcal{I}(y_1,\dots,y_n)$.

The key result of this Note is:

Proposition 1. *$\mathfrak{g}$ has a very nilpotent basis if and only if $\mathfrak{g}$ is nilpotent.*

Proposition 1 can be seen as a refinement of Engel's Theorem (see, e.g., [5, 19.3.6]). Its proof is rather technical and is given in Section 2.

Assume now that $\mathfrak{g}$ is semisimple. Let G be the algebraic adjoint group of $\mathfrak{g}$ and let $p_1,\dots,p_d$ be algebraically independent homogeneous generators of $S(\mathfrak{g}^*)^G$, the set of G-invariant elements of $S(\mathfrak{g}^*)$. It is well known that $\mathcal{V}(p_1,\dots,p_d)$ is the nilpotent cone of $\mathfrak{g}$ (see, e.g., [5, §31]). Let J_0 be the ideal of $S((\mathfrak{g}^n)^*)=\bigotimes_n S(\mathfrak{g}^*)$ generated by the polynomials $p_i\circ f$ where $f\in\mathcal{I}_n$. We define J in the same way, adding the constraint *dep* $f\geqslant 2$. We consider the diagonal action of G on $\mathfrak{g}^n$ and we have $J\subset S((\mathfrak{g}^n)^*)^G$. In Section 3, we show how Proposition 1 implies the following proposition.

Proposition 2.

$$\mathcal{V}(J)=G.(\mathfrak{b}\times\cdots\times\mathfrak{b}),\qquad \mathcal{V}(J_0)=G.(\mathfrak{n}\times\cdots\times\mathfrak{n})$$

where $\mathfrak{b}$ is any Borel subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n}$.

The question of finding such ideals arises naturally when one studies the diagonal action of G on $\bigtimes_{i=1}^n\mathfrak{g}$. Indeed, when $n=1$, $G.\mathfrak{n}$ is the nilpotent cone $\mathcal{N}$. In the $n=2$ case, several authors pointed out nice generalizations of $\mathcal{N}$. Let us mention the set of *nilpotent pairs* of [3], whose *principal* elements lie in a finite number of orbits under the diagonal action of G. Looking at couples of commuting elements, we get the *nilpotent commuting variety* $\mathfrak{C}^{nil}(\mathfrak{g}):=\{(x,y)\in\mathcal{N}\times\mathcal{N}\mid[x,y]=0\}$ studied in [1,4] and which has the nice property of being equidimensional. Finally, the *nilpotent bicone*, studied in [2], is the affine subscheme $\mathfrak{N}\subset\mathfrak{g}\times\mathfrak{g}$ defined by the polarized polynomials $p_i(x+ty)=0,\forall t\in\Bbbk$. Its underlying set consists of pairs whose any linear combination is nilpotent. It is a non-reduced complete intersection, which contains $\mathfrak{C}^{nil}(\mathfrak{g})$ and which has $G.(\mathfrak{n}\times\mathfrak{n})$ as an irreducible component.

2. Very nilpotent basis

The aim of this section is to prove Proposition 1. As a first step, we assume that $\mathfrak{g}$ is semisimple. We are going to prove that $\mathfrak{g}$ has no very nilpotent basis, cf. Corollary 5.

First, we have to state some properties of the characteristic grading of a nilpotent element. Let y be a nilpotent element of $\mathfrak{g}$ and embed y in an $\mathfrak{sl}_2$-triple (y,h,f). Consider the characteristic grading

$$\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(h,i)$$

where $\mathfrak{g}(h,i)=\{z\in\mathfrak{g}\mid[h,z]=iz\}$ and denote by pr_i the projection $\mathfrak{g}\to\mathfrak{g}(h,i)$ with respect to this grading. Then $\mathfrak{g}(h,0)=\mathfrak{g}^h$ is a subalgebra of $\mathfrak{g}$, reductive in $\mathfrak{g}$, i.e. $\mathrm{ad}_{\mathfrak{g}(h,0)}(\mathfrak{g})$ is a semisimple representation.

Lemma 3.

i) *An element $x\in\bigoplus_{i\geqslant 0}\mathfrak{g}(h,i)$ is nilpotent if and only if $\mathrm{pr}_0(x)$ is nilpotent in $\mathfrak{g}(h,0)$.*
ii) *For all i, $\langle[\mathfrak{g}(h,i),\mathfrak{g}(h,-i)]\rangle\subset\mathfrak{g}(0,h)$ is a Lie subalgebra, reductive in $\mathfrak{g}(0,h)$.*

Proof. i) This follows from the fact that $\bigoplus_{i\geqslant 0}\mathfrak{g}(h,i)$ is a parabolic subalgebra of $\mathfrak{g}$ having $\mathfrak{g}(h,0)$ as Levi factor.

ii) If $i=0$, the result is straightforward. In the following, we assume $i\neq 0$. Embed h in a Cartan subalgebra $\mathfrak{h}$. This gives rise to a root system $R(\mathfrak{g},\mathfrak{h})\subset\mathfrak{h}^*$. Choose a fundamental basis B of the root system $R(\mathfrak{g},\mathfrak{h})$ such that h lies in the positive Weyl Chamber associated to B.

Let $[x_1,y_1]$, $[x_2,y_2]$ be two elements of $[\mathfrak{g}(h,i),\mathfrak{g}(h,-i)]$. Then,

$$\big[[x_1,y_1],[x_2,y_2]\big]=\big[\big[[x_1,y_1],x_2\big],y_2\big]+\big[x_2,\big[[x_1,y_1],y_2\big]\big].$$

Since x_2 and $[[x_1,y_1],x_2]$ (resp. y_2 and $[[x_1,y_1],y_2]$) are elements of $\mathfrak{g}(h,i)$ (resp. $\mathfrak{g}(h,-i)$), the element $[[x_1,y_1],[x_2,y_2]]$ belongs to $\langle[\mathfrak{g}(h,i),\mathfrak{g}(h,-i)]\rangle$. By linearity, we deduce that $\langle[\mathfrak{g}(h,i),\mathfrak{g}(h,-i)]\rangle$ is a Lie subalgebra of $\mathfrak{g}(0,h)$.

Write $R_i:=\{\alpha\in R(\mathfrak{g},\mathfrak{h})\mid\alpha(h)=i\}$. Hence $\mathfrak{g}(h,i)=\bigoplus_{i\in R_i}\mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha$ is the root space associated to α. Then, we see that

$$\bigoplus_{R_{i,-i}} \mathfrak{g}^\alpha \subset \big\langle \big[\mathfrak{g}(h,i), \mathfrak{g}(h,-i)\big]\big\rangle \subset \mathfrak{h} \oplus \bigoplus_{R_{i,-i}} \mathfrak{g}^\alpha, \tag{1}$$

where $R_{i,-i} = (R_i + R_{-i}) \cap R(\mathfrak{g},\mathfrak{h}) = (R_i - R_i) \cap R(\mathfrak{g},\mathfrak{h})$. Since $R_{i,-i} = -R_{i,-i}$, $\langle[\mathfrak{g}(h,i), \mathfrak{g}(h,-i)]\rangle$ is reductive and it follows from (1) that its central elements are semisimple in $\mathfrak{g}$. Hence the result. $\square$

The main step of the proof of Proposition 1 lies in the following lemma.

Lemma 4. *If $\mathfrak{g}$ has a very nilpotent basis, then there exists a non-zero proper Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$, reductive in $\mathfrak{g}$, and a basis $(x_1, \ldots, x_p)$ of $\mathfrak{k}$ such that ad_z is nilpotent in $\mathfrak{gl}(\mathfrak{g})$ for each $z \in \mathcal{I}(x_1, \ldots, x_p)$.*

Proof. Let $(y_1, \ldots, y_n)$ be a very nilpotent basis of $\mathfrak{g}$. Each y_i is nilpotent. Embed y_1 in an $\mathfrak{sl}_2$-triple (y_1, h, f) and consider the characteristic grading

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(h,i).$$

Denote by i_0 the highest weight in this decomposition, i.e. $\mathfrak{g}(h,i_0) \neq \{0\}$ and $\mathfrak{g}(h,i) = \{0\}$ for all $i > i_0$. We set $\mathfrak{k} := \langle[\mathfrak{g}(h,i_0), \mathfrak{g}(h,-i_0)]\rangle$. Since $\mathfrak{k} \subset \mathfrak{g}(0,h) \neq \mathfrak{g}$ is reductive in $\mathfrak{g}$ (Lemma 3), there remains to construct the basis $(x_1, \ldots, x_p)$.

The endomorphism ad_{y_1} is nilpotent of order $i_0 + 1$ and

$$(\mathrm{ad}_{y_1})^{i_0} : \mathfrak{g} \to \mathfrak{g}(h,i_0)$$

is surjective. We define $z_j := \mathrm{ad}_{y_1}^{i_0}(y_j)$ for $j \in J := [\![1,n]\!]$. By construction, $z_j \in \mathcal{I}(y_1, \ldots, y_n)$ and $(z_j)_{j\in J}$ is a family spanning the vector space $\mathfrak{g}(h,i_0)$. On the opposite side, we define $y'_k := \mathrm{pr}_{-i_0}(y_k)$ for $k \in J$. The family $(y'_k)_{k\in J}$ spans the vector space $\mathfrak{g}(h,-i_0)$.

Consider now $\mathrm{ad}_{z_j} : \mathfrak{g} \to \bigoplus_{i\geqslant 0} \mathfrak{g}(h,i)$ and define

$$x_{j,k} := \mathrm{pr}_0 \circ \mathrm{ad}_{z_j}(y_k) = \mathrm{ad}_{z_j} \circ \mathrm{pr}_{-i_0}(y_k), \quad j,k \in J^2.$$

The family $(x_{j,k})_{j,k}$ spans the vector space $\mathfrak{k} \subset \mathfrak{g}(h,0)$. The elements $\mathrm{ad}_{z_j}(y_k)$ belong to $\mathcal{I}(y_1, \ldots, y_n)$. Hence, it follows from Lemma 3 that the elements $x_{j,k}$ and their iterated brackets are nilpotent. In other words, if $(x_1, \ldots, x_p)$ is a basis of $\mathfrak{k}$ extracted from $(x_{j,k})_{j,k\in J^2}$, then it fulfills the required properties. $\square$

Corollary 5. *Let $\mathfrak{g} \neq \{0\}$ be a semisimple Lie algebra, then $\mathfrak{g}$ has no very nilpotent basis.*

Proof. We argue by induction on $\dim\mathfrak{g}$. Assume that there is no semisimple $\mathfrak{w} \subset \mathfrak{g}$ such that $0 \neq \dim\mathfrak{w} < \dim\mathfrak{g}$ having a very nilpotent basis. Assume that $\mathfrak{g}$ has one. Then, we define the reductive subalgebra $\mathfrak{k}$ equipped with the basis $(x_1, \ldots, x_p)$ as in Lemma 4. By hypothesis $\mathfrak{k}$ is not semisimple. Hence $\mathfrak{k}$ has a non-trivial center whose elements are semisimple. Therefore there must be some $i \in [\![1,p]\!]$ such that x_i is not nilpotent and we get a contradiction. $\square$

We are now ready to finish the proof of Proposition 1 in the general case. From now on, we forget the semisimplicity assumption on $\mathfrak{g}$.

First of all, we note that whenever $\mathfrak{g}$ is nilpotent, then any basis of $\mathfrak{g}$ is very nilpotent.

Conversely, assume that $\mathfrak{g}$ is any Lie algebra having a very nilpotent basis $(y_1, \ldots, y_n)$. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. The algebra $\mathfrak{g}/\mathfrak{r}$ is semisimple and we can extract a very nilpotent basis $(x_1, \ldots, x_p)$ of $\mathfrak{g}/\mathfrak{r}$ from the projection of the elements y_i via $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$. It follows from Corollary 5 that $\mathfrak{g}/\mathfrak{r} = \{0\}$. In other words $\mathfrak{g}$ is solvable.

Then, one may apply Lie's Theorem (see, e.g., [5, 19.4.4]). It states that $\mathrm{ad}_\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ can be seen as a subspace of a set of upper triangular matrices. In fact, the nilpotency condition on the ad_{y_i} implies that $\mathrm{ad}_\mathfrak{g}$ is a subspace of a set of strictly upper triangular matrices. In particular, $\mathfrak{g}$ is nilpotent. This ends the proof of Proposition 1.

3. *n*-tuples lying in a same Borel subalgebra

In this section $\mathfrak{g}$ is assumed to be semisimple. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. Define two ideals of $S((\mathfrak{g}^n)^*)^G$ by:

$$J_0 := \big(p_i \circ f \;\big|\; i \in [\![1,d]\!],\ f \in \mathcal{I}_n\big), \qquad J := \big(p_i \circ f \;\big|\; i \in [\![1,d]\!],\ f \in \mathcal{I}_n,\ dep\, f \geqslant 2\big),$$

where $p_1, \ldots, p_d$ are as in the introduction.

Proposition 6.

$$\mathcal{V}(J_0) = G.(\mathfrak{n} \times \cdots \times \mathfrak{n}),$$

$$\mathcal{V}(J) = G.(\mathfrak{b} \times \cdots \times \mathfrak{b}).$$

Proof. The inclusions $\mathcal{V}(J_0) \supset G.(\mathfrak{n} \times \cdots \times \mathfrak{n})$ and $\mathcal{V}(J) \supset G.(\mathfrak{b} \times \cdots \times \mathfrak{b})$ are straightforward.

Let us prove the reverse inclusions. Let $(y_1, \ldots, y_n) \in \mathfrak{g}$ and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie subalgebra generated by the elements $(y_1, \ldots, y_n)$. Assume that $(y_1, \ldots, y_n) \subset \mathcal{V}(J_0)$. Choose a basis of $\mathfrak{k}$, $(z_1, \ldots, z_p) \in (\mathcal{I}(y_1, \ldots, y_n))^p$. Then $(z_1, \ldots, z_p)$ is a very nilpotent basis of $\mathfrak{k}$. It follows from Proposition 1 that $\mathfrak{k}$ is nilpotent. Hence there exists $g \in G$ such that $g.\mathfrak{n} \supset \mathfrak{k}$ and $(y_1, \ldots, y_n) \in g.(\mathfrak{n} \times \cdots \times \mathfrak{n})$.

Assume now that $(y_1, \ldots, y_n) \in \mathcal{V}(J)$. Arguing along the same lines, one finds that $[\mathfrak{k}, \mathfrak{k}]$ is nilpotent. Hence $\mathfrak{k}$ is solvable and there exists $g \in G$ such that $g.\mathfrak{b} \supset \mathfrak{k}$. $\square$

The ideals J_0 and J are defined by making use of an infinite number of generators. In fact, if one is more careful with the arguments of Section 2, it is possible to restrict to a finite number. Let us sketch the proof of this. A rough estimation shows that, for $(y_1, \ldots, y_n) \in \mathfrak{g}^n$, the subalgebra $\mathfrak{k}$ generated by $(y_1, \ldots, y_n)$ is spanned by $\{f(y_1, \ldots, y_n) \mid f \in \mathcal{I}_n,\ dep\, f \leqslant \dim \mathfrak{g}\}$ as a vector space. Then, assume that $(z_1, \ldots, z_p)$ is a basis of a semisimple Lie algebra $\mathfrak{k}$. One can restrict in the proof of Lemma 4 and Corollary 5 to the assumption that $f(z_1, \ldots, z_p)$ is nilpotent for $dep\, f \leqslant \dim \mathfrak{k}$. Defining

$$J_0 := \big(p_i \circ f \;\big|\; dep\, f \leqslant (\dim \mathfrak{g})^2\big), \qquad J := \big(p_i \circ f \;\big|\; 2 \leqslant dep\, f \leqslant 2(\dim \mathfrak{g})^2\big),$$

we claim that:

Claim 7. *$\mathcal{V}(J') = \mathcal{V}(J)$ and $\mathcal{V}(J'_0) = \mathcal{V}(J_0)$.*

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