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Harmonic Analysis/Dynamical Systems

On systems of dilated functions

Sur les systèmes de fonctions dilatées

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ABSTRACT

If $f(x)=\sum_{\ell\in\mathbb{Z}}a_\ell e^{2i\pi\ell x}$ satisfies $\sum_{\nu\geqslant 1}a_\nu^2\Delta(\nu)<\infty$, where Δ is the Erdös–Hooley function, we show that the series $\sum_{k=0}^\infty c_k f(kx)$ converges for almost every x, whenever the coefficient sequence verifies the condition

$$\sum_r\left(\sum_{j=2^r+1}^{2^{r+1}}c_j^2 d(j)(\log j)^2\right)^{1/2}<\infty,$$

d being the divisor function. This strongly improves earlier related results.

RÉSUMÉ

Pour toute fonction $f(x)=\sum_{\ell\in\mathbb{Z}}a_\ell e^{2i\pi\ell x}$ telle que $\sum_{\nu\geqslant 1}a_\nu^2\Delta(\nu)<\infty$, où Δ est la fonction de Erdös–Hooley, nous montrons que la série $\sum_{k=0}^\infty c_k f(kx)$ converge presque partout dès que la suite des coefficients vérifie

$$\sum_r\left(\sum_{j=2^r+1}^{2^{r+1}}c_j^2 d(j)(\log j)^2\right)^{1/2}<\infty,$$

$d(n)$ désignant la fonction des diviseurs de n. Ceci améliore considérablement un certain nombre de résultats partiels précédemment obtenus.

1. Introduction – main result

Let $\mathbb{T}=\mathbb{R}/\mathbb{Z}\simeq[0,1[$. Let $e(x)=\exp(2i\pi x)$, $e_n(x)=e(nx)$, $n\in\mathbb{Z}$. Let $f(x)=\sum_{\ell\in\mathbb{Z}}a_\ell e_\ell$, $a_{-\nu}=a_\nu$, $a_0=0$, $\underline{a}=\{a_k,k\geqslant 0\}\in\ell^2(\mathbb{N})$. We denote throughout $f_n(x)=f(nx)$. Assume $f\in\mathrm{Lip}_\alpha(\mathbb{T})$ for $\alpha>1/2$. The convergence properties of the system $\{f_n,n\geqslant 1\}$ were studied by many authors, among them, Erdös, Kac, Khintchin, Koksma, Marstrand, Wintner and more recently Berkes, Gaposhkin, Nikishin, ... See [2], Chapter 2 to which we also refer for convenience, concerning all results quoted in this Note. Using Carleson's theorem, Gaposhkin [4] has showed that if $f\in\mathrm{Lip}_\alpha(\mathbb{T})$, $\alpha>1/2$, the series $\sum_{k=0}^\infty c_k f_k(x)$ converges for almost all $x\in\mathbb{T}$, for any $\underline{c}=\{c_k,k\geqslant 0\}\in\ell^2(\mathbb{N})$. Berkes [1] showed that this result becomes false if $f\in\mathrm{Lip}_{1/2}(\mathbb{T})$. For $f\in\mathrm{Lip}_\alpha(\mathbb{T})$ with $0<\alpha<1/2$, only very partial results exist. The purpose of this Note is to establish a

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quite sharp result valid for considerably much larger classes of functions. Our approach is not based on Carleson's theorem. Let $d(k) = \sharp\{d\colon\ d|k\}$ be the divisor function. Introduce the Erdös–Hooley function

$$\Delta(\nu) = \sup_{u\in\mathbb{R}} \sum_{\substack{d|\nu\\ u<d\leqslant eu}} 1.$$

Theorem 1.1. *Assume that $A = \sum_{\nu\geqslant 1} a_\nu^2 \Delta(\nu) < \infty$ and $B = \sum_{r\geqslant 1} (\sum_{j=2^r+1}^{2^{r+1}} c_j^2 d(j)(\log j)^2)^{1/2} < \infty$. Then the series $\sum_{k=0}^\infty c_k f_k(x)$ converges for almost all $x \in \mathbb{T}$.*

Some comments are necessary. It is well known that the arithmetical properties of the support of $\underline{c}$ play a crucial role in the study of this problem. Our series condition B reflects this fact in a very simple way. In particular, if $\sup\{d(\nu),\ \nu \in \operatorname{supp}\{\underline{c}\}\} < \infty$, condition B reduces to $B' = \sum_{r\geqslant 1} (\sum_{j=2^r+1}^{2^{r+1}} c_j^2 (\log j)^2)^{1/2} < \infty$, which is realized once $\sum_j c_j^2 (\log j)^b < \infty$ for some $b > 3$. Let $\{p_j,\ j \geqslant 1\}$ be a sequence of prime numbers. As a consequence, we obtain that the series $\sum_{k=0}^\infty \gamma_k f(p_k x)$ converges a.e. whenever $\sum_k \gamma_k^2 (\log p_k)^b < \infty$, $b > 4$ and f verifies the mild condition $A < \infty$. Concerning this case, we don't know comparable results, see however [2], Corollaries 2.3, 2.3*, as well as Theorem 3.2. Notice also that replacing $d(j)$ by its classical majorant [5], $d(j) = \mathcal{O}(c_0^{\log j/\log\log j})$, $c_0 > 2$, in the definition of B, would provide in general a much weaker result. These considerations yield the importance of the presence of the factor $d(j)$. Now consider other applications.

Assume that $a_\nu = \mathcal{O}(\nu^{-\alpha})$, $\alpha > 1/2$. As $\Delta(\nu) \leqslant d(\nu) \ll_\varepsilon \nu^\varepsilon$, it follows that $A < \infty$. Moreover $B < \infty$ once $\sum_k c_k^2 k^\varepsilon < \infty$ for some $\varepsilon > 0$. This improves Corollary 2.5* in [2], where it was assumed that $\sum_k c_k^2 k^{1-\alpha} (\log k)^2 < \infty$.

It is trivial that condition A is fulfilled if $f \in \operatorname{Lip}_\alpha(\mathbb{T})$ for some $\alpha > 0$, since in this case $\sum_{2^s<j\leqslant 2^{s+1}} a_j^2 \leqslant C2^{-2s\alpha}$ and $\Delta(\nu) \ll_\varepsilon \nu^\varepsilon$. Thus the series $\sum_{k=0}^\infty c_k f_k(x)$ converges almost everywhere in particular if $\sum_k c_k^2 k^\varepsilon < \infty$ for some $\varepsilon > 0$. This improves when $0 < \alpha < 1/2$ an earlier result due to Gaposhkin, where it was assumed that $\sum_k c_k^2 k^{1-2\alpha} (\log k)^{2\beta} < \infty$ for some $\beta > 1 + 2\alpha$.

According to Theorem 2B of [6], $\sum_{\nu\leqslant y} \Delta(\nu) = \mathcal{O}(y \log^{\frac{4}{\pi}-1} y)$. As $4/\pi - 1 \approx 0,27324$, Δ has a comparatively slower mean behavior than d since as is well known, $\sum_{\nu\leqslant y} d(\nu) \sim y\log y$. Using partial summation, we see that condition $A < \infty$ is also fulfilled once $A' = \sum_{\nu=1}^\infty \nu |a_{\nu-1}^2 - a_\nu^2| \log^{\frac{4}{\pi}-1} \nu < \infty$. And this reduces to $\sum_{\nu=1}^\infty a_\nu^2 \log^{4/\pi-1} \nu < \infty$, if $\underline{a}$ is monotonic.

2. Proof of Theorem 1.1

The introduction of the Erdös–Hooley function $\Delta(\nu)$ for these questions turns up to be very appropriate. Indeed, it allows us to propose a surprisingly simple proof. We will use the fact [6, p. 119] that for all positive integers u and ν, $\Delta(u\nu) \leqslant d(u)\Delta(\nu)$. Given any set K of positive integers, we denote $d(K,n) = \sharp\{d \in K : d|n\}$. By using Plancherel formula, next Cauchy–Schwarz's inequality,

$$\Big\|\sum_{k\in K} c_k f_k\Big\|_2^2 = \sum_{n=1}^\infty \Big(\sum_{\substack{k|n\\ k\in K}} a_{\frac{n}{k}} c_k\Big)^2 \leqslant \sum_{n=1}^\infty \Big(\sum_{\substack{k|n\\ k\in K}} a_{\frac{n}{k}}^2 c_k^2\Big) d(K,n) = \sum_{k\in K} c_k^2 \sum_{\nu=1}^\infty a_\nu^2 d(K,\nu k). \tag{1}$$

Let $K \subset]e^r, e^{r+1}]$. Then,

$$\Big\|\sum_{k\in K} c_k f_k\Big\|_2^2 \leqslant \sum_{k\in K} c_k^2 \Big(\sum_{\nu=1}^\infty a_\nu^2 d\big(]e^r, e^{r+1}], \nu k\big)\Big) \leqslant \sum_{k\in K} c_k^2 \Big(\sum_{\nu=1}^\infty a_\nu^2 \Delta(\nu k)\Big) \leqslant \Big(\sum_{\nu=1}^\infty a_\nu^2 \Delta(\nu)\Big) \sum_{k\in K} c_k^2 d(k).$$

Put $X_j = \sum_{u=1}^j c_u f_u$, $t_j = B^{-1} \sum_{u=1}^j c_u^2 d(u)$. Thus $\|X_j - X_i\|_2 \leqslant (AB)^{1/2} (t_j - t_i)^{1/2}$, $e^r < i \leqslant j < e^{r+1}$. Using Lemma 8.3.4 from [7] for instance, we deduce that $\|\sup_{2^r<\ell<k\leqslant 2^{r+1}} |X_k - X_\ell|\|_2 \leqslant CBr$. Thereby,

$$\Big\|\sup_{2^r<\ell<k\leqslant 2^{r+1}} \Big|\sum_{j=\ell+1}^k c_j f_j\Big|\Big\|_2 \leqslant CB\Big(\sum_{2^r<j\leqslant 2^{r+1}} c_j^2 d(j)\Big)^{1/2} r \leqslant CB\Big(\sum_{2^r<j\leqslant 2^{r+1}} c_j^2 d(j)(\log j)^2\Big)^{1/2}. \tag{2}$$

Now we can finish the proof using a classical scheme. If $S \geqslant R$ and $2^R < k < 2^{S+1}$, then

$$\Big|\sum_{j=2^R+1}^k c_j f_j\Big| \leqslant \sum_{R\leqslant r\leqslant S} \sup_{2^r<h\leqslant 2^{r+1}} \Big|\sum_{j=2^r+1}^h c_j f_j\Big|.$$

Hence,

$$\Big\|\sup_{k>2^R}\Big|\sum_{j=2^R+1}^{k} c_j f_j\Big|\Big\|_2 \leqslant \Big\|\sum_{r\geqslant R}\sup_{2^r<k\leqslant 2^{r+1}}\Big|\sum_{j=2^r+1}^{k} c_j f_j\Big|\Big\|_2 \leqslant \sum_{r\geqslant R}\Big\|\sup_{2^r<k\leqslant 2^{r+1}}\Big|\sum_{j=2^r+1}^{k} c_j f_j\Big|\Big\|_2$$
$$\leqslant C\sum_{r\geqslant R}\Bigg(\sum_{j=2^r+1}^{2^{r+1}} c_j^2 d(j)(\log j)^2\Bigg)^{1/2}.$$

Therefore, by the assumptions made, the oscillation of the sequence $\{\sum_{j=1}^{k} c_j f_j, k\geqslant 1\}$ tends to zero almost everywhere. This achieves the proof.

Final remarks. Suppose that $\underline{a}$, $\underline{c}$ have mutually coprime supports. If $K\subset \mathrm{support}(\underline{c})$, $\nu\in\mathrm{support}(\underline{a})$, then $d(K,\nu k)=d(K,k)$, and so (1) becomes $\|\sum_{k\in K} c_k f_k\|_2^2 \leqslant \|f\|_2^2\sum_{k\in K} c_k^2 d(K,k)$. By arguing similarly, we also deduce that if $B'=\sum_{r\geqslant 1}(\sum_{j=2^r+1}^{2^{r+1}} c_j^2\Delta(j)(\log j)^2)^{1/2}<\infty$, then the series $\sum_{k=0}^{\infty} c_k f_k(x)$ converges a.e. Although we did not appeal to Carleson's theorem, it is worth observing that one can always remove from f its "Carleson" component. Let indeed $f^\flat=\sum_{|a_\ell|>\varepsilon_\ell} a_\ell e_\ell$ and assume that $A'=\sum_{|a_\ell|>\varepsilon_\ell}|a_\ell|^2/\varepsilon_\ell<\infty$. Plainly,

$$\sup_{V\leqslant u\leqslant v\leqslant W}\Big|\sum_{u\leqslant n\leqslant v} c_n f_{k_n}^\flat\Big| \leqslant \sum_\ell |a_\ell^\flat| \sup_{V\leqslant u\leqslant v\leqslant W}\Big|\sum_{u\leqslant n\leqslant v} c_n e_{\ell k_n}\Big|.$$

By integrating, next using Carleson–Hunt's theorem [3], we get

$$\Big\|\sup_{V\leqslant u\leqslant v\leqslant W}\Big|\sum_{n=u}^{v} c_n f_{k_n}^\flat\Big|\Big\|_2 \leqslant \sum_\ell |a_\ell^\flat| \Big\|\sup_{V\leqslant u\leqslant v\leqslant W}\Big|\sum_{u\leqslant n\leqslant v} c_n e_{\ell k_n}\Big|\Big\|_2 \leqslant CA\sum_{k=V}^{W} c_k^2.$$

Therefore, the sequence $\{\sum_{n=1}^{N} c_n f_{k_n}^\flat, N\geqslant 1\}$ has oscillation near infinity tending to zero a.e. In other words, the series $\sum_n c_n f_{k_n}^\flat$ converges a.e.

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