



Algebraic Geometry

Family of counterexamples to King's conjecture <sup>☆</sup>*Famille de contre-exemples à la conjecture de King*Mateusz Michałek <sup>a,b</sup><sup>a</sup> *Mathematical Institute of the Polish Academy of Sciences, Św. Tomasza 30, 31-027 Kraków, Poland*<sup>b</sup> *Institut Fourier, universite Joseph-Fourier, 100, rue des Maths, BP 74, 38402 St Martin d'Hères, France*

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## ABSTRACT

In this short Note we present an infinite family of arbitrary high dimensional counterexamples to the King's conjecture.

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## R É S U M É

Dans cette Note nous présentons une famille infinie de contre-exemples à la conjecture de A. King, de dimension arbitrairement grande.

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## 1. Introduction

In [6] King made the following conjecture:

**Conjecture 1.1.** *For any smooth, complete toric variety  $X$  there exists a full, strongly, exceptional collection of line bundles.*

The conjecture turned out to be false. The first counterexample was given by Hille and Perling, in [4]. They showed a smooth, complete toric surface that does not have a full strongly exceptional collection of line bundles. Although very nice techniques were used, the proof was burdensome and required large tables of cases. Later counterexamples of higher dimension were found and the case of toric surfaces was classified [5]. The main purpose of this Note is to present a rather compact, combinatorial proof that an infinite family of higher dimensional varieties does not satisfy King's conjecture. The varieties we consider were suspected to be counterexamples and these are  $\mathbb{P}^n$  blown up in two points. It turns out that for  $n$  sufficiently large the longest strongly exceptional collection of line bundles is (by a factor) shorter than the rank of the Grothendieck group, hence such a collection is far from being full.

For more information on this topic the reader is advised to look in [2,3] and references therein.

## 2. Notation

The varieties we consider are of Picard number 3. Such (smooth, complete, toric) varieties have been fully classified by Batyrev in [1] in terms of their *primitive collections*. Using this classification  $\mathbb{P}^n$  blown up in two points is given by  $|X_0| = |X_2| = |X_3| = |X_4| = 1$  and  $|X_1| = n - 1$  with all other parameters equal to 0. Choosing the basis of the one parameter

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subgroups lattice  $N$  equal to  $v_1, y_2, \dots, y_{n-1}, z_1$  the ray generators of the fan are the basis elements and vectors  $y_1, t_1, u_1$  satisfying:

$$t_1 = -z_1, \quad y_1 = -y_2 - \dots - y_{n-1} - z_1 - v_1, \quad u_1 = -v_1.$$

The rank of the Grothendieck group is equal to the number of maximal cones that is  $3n - 1$ . To each element of  $X_i$  corresponds a  $T$ -divisor. All divisors in a given  $X_i$  are linearly equivalent and are given by  $D_v, D_y, D_z, D_t, D_u$  respectively for  $i = 0, 1, 2, 3, 4$ .

In the strongly exceptional collection, differences of divisors cannot have nonzero higher cohomology. Divisors with nonzero higher cohomology will be called forbidden. The following classification of forbidden divisors is very easy to establish. In a general case of Picard number three this has been done in [7], but in this special case one can use arguments of elementary topology. The forbidden divisors in our case are  $\alpha_1 D_v + \alpha_2 D_y + \alpha_3 D_z + \alpha_4 D_t + \alpha_5 D_u$ , where exactly 2, 3 or 5 consecutive (in a cyclic way, that is indices are considered modulo 5)  $\alpha$ 's are negative and if  $\alpha_2 < 0$ , then  $\alpha_2 \leq -n + 1$ .

We have  $D_z = D_t + D_y$  and  $D_v = D_u + D_y$ . We choose the basis  $D_y, D_t, D_u$ , what gives us forbidden divisors  $(\alpha_1 + \alpha_2 + \alpha_3)D_y + (\alpha_3 + \alpha_4)D_t + (\alpha_1 + \alpha_5)D_u$  with the conditions on  $\alpha$ 's as above. A divisor  $aD_y + bD_t + cD_u$  will be denoted by  $(a, b, c)$  and we reserve precise letters for precise coordinates. A line bundle  $L_1$  will be called compatible with  $L_2$  iff they can both appear in a strongly exceptional collection, that is iff  $L_1 - L_2$  and  $L_2 - L_1 = -(L_1 - L_2)$  are not forbidden.

### 3. Proof

Let us fix a strongly exceptional collection  $E$ . We assume without loss of generality that  $0 \in E$  and that all other divisors in  $E$  have nonnegative coefficient  $a$ .

**Lemma 3.1.** *The only divisors with  $a = 0$  compatible with  $(0, 0, 0)$  are:*

$$(0, -1, 0), \quad (0, 0, -1), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, -1, 1), \quad (0, 1, -1).$$

**Proof.** If  $b < -1$ , then we take  $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = -1, \alpha_4$ -negative to obtain  $b, \alpha_5$ -any to obtain  $c$ . Analogously for  $c < -1$ , hence  $-1 \leq b, c \leq 1$ . Moreover  $(0, -1, -1)$  is also bad (so also  $(0, 1, 1)$ ).  $\square$

**Corollary 3.2.** *There can be at most 3 distinct line bundles with  $a = 0$  in  $E$ . For a fixed  $a$  we can have only 3 line bundles in  $E$ .*

**Proof.** Follows by inspection.  $\square$

**Lemma 3.3.** *For  $a > 0$  the only line bundles  $(a, b, c)$  that are not forbidden must satisfy  $-1 \leq b \leq a$  and  $-1 + a - b \leq c \leq a$  (and by symmetry  $-1 + a - c \leq b \leq a$ ).*

**Proof.** For  $b < -1$  we take  $\alpha_1 = 0, \alpha_3 = -1, \alpha_2$ -positive to have  $a, \alpha_4$ -negative to have  $b, \alpha_5$ -any to have  $c$ . For  $b > a$  we look at  $(-a, -b, -c)$  and take  $\alpha_3 = -a, \alpha_1 = \alpha_2 = 0, \alpha_4$ -negative to have  $-b, \alpha_5$ -any to have  $-c$ . In the same way  $-1 \leq c \leq a$ .<sup>1</sup> So the only case that we have to exclude is  $-1 \leq c < -1 + a - b$ . In such a case we can take  $\alpha_4 = -1, \alpha_3 = b + 1, \alpha_2 = 0, \alpha_1 = a - b - 1, \alpha_5 = c - a + b + 1 < 0$ .  $\square$

**Lemma 3.4.** *For three consecutive parameters  $a$ 's there can be at most 8 line bundles in  $E$ .*

**Proof.** We assume without loss of generality  $0 \leq a \leq 2$ . If the lemma does not hold, then from Corollary 3.2 we would have to have 3 line bundles for each  $a$ . For  $a = 0$  we can have either:

Case 1:  $(0, 0, 0), (0, -1, 0), (0, 0, -1)$  then for  $a = 1$  there is only one compatible from Lemma 3.3 namely  $(1, 0, 0)$ .

Case 2:  $(0, 0, 0), (0, 1, 0), (0, 0, 1)$  then for  $a = 1$  the compatible line bundles are  $(1, 1, 1), (1, 1, 0), (1, 0, 1)$ . If we choose all of them then the only one compatible for  $a = 2$  is  $(2, 1, 1)$  from Lemma 3.3.

Case 3:  $(0, 0, 0), (0, -1, 0), (0, -1, 1); (0, 0, 0), (0, 0, -1), (0, 1, -1); (0, 0, 0), (0, 1, 0), (0, 1, -1); (0, 0, 0), (0, 0, 1), (0, -1, 1)$ . All these possibilities are cases 1 or 2 after subtracting a divisor from all three considered divisors.  $\square$

**Definition 3.5.** Line bundles in the collection  $E$  with  $a > n$  are called high. Others are called low.

**Lemma 3.6.** *A high line bundle is forbidden unless either  $b = 1$  (high bundles of type 1) or  $c = 1$  (high bundles of type 2).*

**Proof.** Suppose that  $b = 0$  or  $b = -1$ . We show that  $(-a, -b, -c)$  is forbidden. Take  $\alpha_1 = -1, \alpha_2 = -a + 1, \alpha_3 = 0, \alpha_4 = -b, \alpha_5$ -any to obtain  $-c$ . So  $b \geq 1$  and analogously  $c \geq 1$ . If both coefficients were strictly greater than 1 we would obtain  $(-a, -b, -c)$  by taking all  $\alpha$ 's negative.  $\square$

<sup>1</sup> The parameters  $b$  and  $c$  are in symmetry.

**Lemma 3.7.** *We cannot have high line bundles of both types in  $E$ .*

**Proof.** From Lemma 3.3 a high line bundle must have the coordinate different from 1 greater or equal to  $n - 1$ . If we subtract two high line bundles of different types we can assume that the first coordinate is positive and one of the others will be less or equal to  $-n + 2$  what contradicts Lemma 3.3 for  $n > 3$ .  $\square$

From now on without loss of generality we assume that we only have high line bundles of type 1 in  $E$ . Let us project all high line bundles from  $E$  onto the first coordinate obtaining a subset of  $\mathbb{N}$ . Suppose that this subset has got  $k$  elements, that is high line bundles can have  $k$  different parameters  $a$ . We obtain:

**Lemma 3.8.** *There are at most  $k + 2$  high line bundles in  $E$ .*

**Proof.** We assumed that  $0 \in E$ , so the high line bundles in  $E$  must not be forbidden. We know that for each high line bundle in  $E$  we have  $b = 1$ , so from Lemma 3.3 we know that  $0 \leq a - c \leq 2$ . Let us notice that the difference  $a - c$  cannot decrease when  $a$  increases for high line bundles in  $E$ . Indeed suppose that we have two high line bundles in  $E$  of the form  $(a_1, 1, c_1)$ ,  $(a_2, 1, c_2)$  with  $a_2 > a_1$  and  $a_2 - c_2 < a_1 - c_1$ . By subtracting these two line bundles we obtain  $(a_2 - a_1, 0, c_2 - c_1)$  that is forbidden by Lemma 3.3.

Notice that each time we have more than one line bundle for a fixed  $a$  then the difference  $a - c$  strictly increases. This means that we can have one line bundle for each  $a$  plus possibly two more as  $a - c$  increases from 0 to 2. This gives us in total  $k + 2$  line bundles.  $\square$

**Proposition 3.9.** *There are at most  $\frac{8}{3}(n - 1) + 6$  low line bundles (from Lemma 3.4), so  $k > 0$  for  $n > 13$ .*

**Remark 3.10.** Of course  $k$  is at most  $n + 1$ . Otherwise we would have two high line bundles in  $E$  with the difference that is high. By Lemma 3.7 the difference would have  $b = 0$ , hence by Lemma 3.6 it would have  $c = 1$  and would be forbidden by Lemma 3.3.

From the definition of  $k$  we know that there is a line bundle  $L = (a, 1, c)$  in  $E$ , with  $a \geq n + k$ . Now we investigate line bundles with  $a < k$ , that are called very low.

**Lemma 3.11.** *Each very low line bundle in  $E$  must have  $b = 0$ .*

**Proof.** Let  $B$  be a very low line bundle.  $L - B$  is high, so from Lemma 3.6 either the second or third coordinate is 1. The third one is  $c_L - c_B \geq a_L - 2 - a_B > n + k - 2 - k = n - 2 > 1$ , for  $n > 3$ . We see that  $b_L - b_B = 1$ . As  $b_L = 1$  the lemma follows.  $\square$

For very low line bundles in  $E$  the parameter  $c$  is either  $a$  or  $a - 1$  by Lemma 3.3 and Lemma 3.11. Reasoning analogously to the proof of Lemma 3.8, we see that there are at most  $k + 1$  very low line bundles (the difference  $a - c$  cannot decrease).

**Theorem 3.12.** *The sequence  $E$  can have at most:  $k + 1 + \frac{8}{3}(n - k - 1) + 6 + k + 2 \leq \frac{8}{3}n - \frac{2}{3}k + \frac{19}{3} < 3n - 1$  for  $n > 20$ .*

**Remark 3.13.** The bounds on  $n$  can be easily improved. For example by considering separately the case  $k = 1$  one can decrease the bound to  $n > 18$ . We concentrated rather on brevity of the proof than sharp bounds.

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## References

- [1] V.V. Batyrev, On the classification of smooth projective toric varieties, *Tōhoku Math. J.* 43 (1991) 569–585.
- [2] L. Costa, R.M. Miró-Roig, Derived category of toric varieties with small Picard number, preprint.
- [3] L. Costa, R.M. Miró-Roig, Tilting sheaves on toric varieties, *Math. Z.* 248 (4) (2004) 849–865.
- [4] L. Hille, M. Perling, A counterexample to King's conjecture, *Compos. Math.* 142 (2006) 1507–1521.
- [5] L. Hille, M. Perling, Exceptional sequences of invertible sheaves on rational surfaces, *Compos. Math.*, arXiv:0810.1936, in press.
- [6] A. King, Tilting bundles on some rational surfaces, preprint at <http://www.maths.bath.ac.uk/~masadk/papers/>.
- [7] M. Lason, M. Michalek, On the derived categories of toric varieties with Picard number three, preprint.