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Functional Analysis

A weak Hilbert space with few symmetries

Un espace failble de Hilbert avec peu de symétries

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ABSTRACT

We construct a separable Banach space $\mathfrak{X}_{wh}$ with an unconditional basis that is a weak Hilbert space and no block subspace is linearly isomorphic to any of its proper subspaces. We prove that the space $\mathfrak{X}_{wh}$ satisfies these properties by showing it is strongly asymptotic ℓ_2 and that every bounded linear operator on $\mathfrak{X}_{wh}$ is a strictly singular perturbation of a diagonal operator with respect to the unit vector basis.

RÉSUMÉ

Nous construisons un space de Banach $\mathfrak{X}_{wh}$ qui est un espace failble de Hilbert et n'admettant aucune sous-espace bloc isomorphe linéaire à une sous-espace. Nous démontrons les propriétés de $\mathfrak{X}_{wh}$ par démontrons que $\mathfrak{X}_{wh}$ est fortement asymptotique ℓ_2 et tout opérateur borné de $\mathfrak{X}_{wh}$ soit une variation strictment singulière d'un opérateur diagonal par rappert à la base.

1. Introduction

A Banach space X is called a *weak Hilbert* space [10] if there are positive constants δ and C such that every finite dimensional subspace E of X contains a subspace F such that $\dim F \geqslant \delta \dim E$, the Banach–Mazur distance between F and $\ell_2^{\dim F}$ is at most equal to C and there is a projection from X to F with norm at most C. It has been shown by G. Pisier [11] that the Fredholm theory, as developed by Grothendieck, works in weak Hilbert spaces. W.B. Johnson [8] showed that the 2-convexification of Tsirelson's space is a weak Hilbert space; thus exhibiting a weak Hilbert space not containing ℓ_2. Other constructions related to the higher order modified Tsirelson spaces could also provide more exotic weak Hilbert spaces.

In this Note we construct an example of a weak Hilbert space $\mathfrak{X}_{wh}$ with an unconditional basis that has a quite asymmetric structure. In particular, every operator on a block subspace Y of $\mathfrak{X}_{wh}$ is a strictly singular perturbation of a diagonal operator on $\mathfrak{X}_{wh}$ restricted to Y. This implies that no block subspace is linearly isomorphic to any of its proper subspaces. In addition, for disjointly supported block subspaces Y and Z of $\mathfrak{X}_{wh}$ every operator $T : Y \to Z$ is strictly singular. Moreover, $\mathfrak{X}_{wh}$ does not contain a quasi minimal subspace and (using the terminology from [6]) its basis is *tight by support*. In this announcement we will outline the argument required to show that every operator on a block subspace has the desired decomposition.

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2. Description of $\mathfrak{X}_{wh}$

The definition of the space $\mathfrak{X}_{wh}$ uses a type of modified mixed Tsirelson saturation method. The space $\mathfrak{X}_{wh}$ is defined as the completion of $c_{00}(\mathbb{N})$ under a norm induced by a norming set of functionals denoted by D_{wh} and described as follows.

For each $n \in \mathbb{N}$ let $\mathcal{S}_n$ denote the Schreier family of order n. See [1,2] for precise definitions. We note that these families contain only finite subsets of $\mathbb{N}$ and satisfy $\mathcal{S}_k \subset \mathcal{S}_n$ for $k < n$. A sequence $(E_i)_{i=1}^d$ of finite subsets of $\mathbb{N}$ is $\mathcal{S}_n$-*admissible* if $E_1 < E_2 < \cdots < E_d$ and $(\min E_i)_{i=1}^d \in \mathcal{S}_n$. A sequence $(E_i)_{i=1}^d$ of finite sets of $\mathbb{N}$ is $\mathcal{S}_n$-*allowable* if $(E_i)_{i=1}^d$ are pairwise disjoint and $(\min E_i)_{i=1}^d \in \mathcal{S}_n$. A sequence of vectors $(x_i)_{i=1}^d$ is $\mathcal{S}_n$-allowable (resp. admissible) if $(\operatorname{supp} x_i)_{i=1}^d$ is $\mathcal{S}_n$-allowable (resp. admissible).

The definition of $\mathfrak{X}_{wh}$ requires that we fix two increasing sequences of positive integers $(n_i)_{i=0}^\infty$ and $(m_i)_{i=0}^\infty$ satisfying certain growth conditions. Let $m_0 = m_1 = 2$, $n_0 = 1$ and for $j \geqslant 2$ let $m_{j+1} \geqslant m_j^3$, $\ell_j = 3\log_2(m_j) + 1$; n_j is chosen such that $\ell_j(n_{j-1} + 1) < n_j$. Now fix infinite disjoint subsets N_1 and N_2 such that $\mathbb{N} = N_1 \cup N_2$. Let,

$$\Sigma = \left\{(E_i, 2j_i)_{i=1}^n:\ E_i \cap E_j = \emptyset \text{ and } j_1 < j_2 < \cdots < j_n \text{ with } j_1 \in N_1 \text{ and } j_i \in N_2 \text{ for } i > 1\right\}.$$

Let $\sigma : \Sigma \to N_2$ be an injection satisfying the following growth condition:

$$m_{2\sigma((E_1,2j_1),\dots,(E_{i+1},2j_{i+1}))} > m_{2\sigma((E_1,2j_1),\dots,(E_i,2j_i))} \cdot (\operatorname{maxsupp} E_i)^2.$$

We need the following two definitions:

Definition 1. Let $D \subset c_{00}(\mathbb{N})$, $m > 1$ and $n \in \mathbb{N}$. We say that D is closed in the modified $\ell_2 - (1/m, \mathcal{S}_n)$ operation if for every $(f_i)_{i=1}^d \subset D$ such that $(f_i)_{i=1}^d$ is $\mathcal{S}_n$-allowable and $(\lambda_i)_{i=1}^d \in Ba(\ell_2)$ the vector

$$\frac{1}{m}\sum_{i=1}^d \lambda_i f_i \in D.$$

Let $\omega(f) = m$ (weight of f) whenever f is the result of the above operation.

Definition 2 *(σ-special sequences).*

(1) A sequence $(E_i, 2j_i)_{i=1}^\infty$ is σ-*special* if $j_1 \in N_1$ and for each $i \geqslant 1$,

$$\sigma\big((E_1, 2j_1), \dots, (E_i, 2j_i)\big) = j_{i+1}.$$

(2) A σ-special sequence $(E_i, 2j_i)_{i=1}^p$ is a $\mathcal{S}_{n_{2j+1}}$ *σ-special sequence* if $(\min E_i)_{i=1}^p \in \mathcal{S}_{n_{2j+1}}$ and $j_1 > j + 1$.

(3) $(f_i)_{i=1}^p \subset c_{00}(\mathbb{N})$ is a *σ-special sequence of functionals* (resp. *$\mathcal{S}_{n_{2j+1}}$ σ-special sequence of functionals*) if there exists a σ-special sequence $(E_i, 2j_i)_{i=1}^p$ (resp. $\mathcal{S}_{n_{2j+1}}$ σ-special sequence) such that $\operatorname{supp} f_i \subset E_i$ and $\omega(f_i) = m_{2j_i}$ for each $1 \leqslant i \leqslant p$.

We are now ready to define D_{wh} as follows:

Definition 3. The norming set D_{wh} is the minimal subset of $c_{00}(\mathbb{N})$ such that

(1) $\{\pm e_n^*:\ n \in \mathbb{N}\} \subset D_{wh}$.
(2) D_{wh} is closed under $\ell_2 - (1/m_{2j}, \mathcal{S}_{n_{2j}})$ operations for all $j \in \mathbb{N}$.
(3) D_{wh} is closed under $\ell_2 - (1/m_{2j+1}, \mathcal{S}_{n_{2j+1}})$ operations for all $j \in \mathbb{N}$ on $\mathcal{S}_{n_{2j+1}}$ σ-special sequences of functionals.

The space $\mathfrak{X}_{wh}$ is the completion of $c_{00}(\mathbb{N})$ under the norm induced by D_{wh}. Namely, for $x \in c_{00}(\mathbb{N})$ let

$$\|x\| = \sup\left\{\left|f(x)\right|:\ f \in D_{wh}\right\}.$$

Finally, note that the norm satisfies

$$\|x\| = \max\left\{\sup\left\{\|x\|_j:\ j \in \mathbb{N} \cup \{0\}\right\}, \|x\|_\infty\right\}$$

where for each $j \in \mathbb{N}$, $\|\cdot\|_j$ satisfies the following implicit formulas:

$$\|x\|_{2j} = \sup\left\{\frac{1}{m_{2j}}\left(\sum_{i=1}^{k}\|E_i x\|^2\right)^{\frac{1}{2}} : (E_i)_{i=1}^{k} \text{ is } \mathcal{S}_{n_{2j}}\text{-allowable}\right\}, \tag{1}$$

$$\|x\|_{2j+1} = \sup\left\{\frac{1}{m_{2j+1}}\left(\sum_{i=1}^{k}\|E_i x\|_{2j_i}^2\right)^{\frac{1}{2}} : (E_i, 2j_i)_{i=1}^{k} \text{ is a } \mathcal{S}_{n_{2j+1}}\ \sigma\text{-special sequence}\right\}. \tag{2}$$

From the above, it is easy to see that the unit vector basis of $\mathfrak{X}_{wh}$ is unconditional.

3. Properties of $\mathfrak{X}_{wh}$

In order to verify that $\mathfrak{X}_{wh}$ is a weak Hilbert space we will use a sufficient condition due to N.J. Nielsen and N. Tomczak-Jaegermann [9]. By applying results of Johnson, they showed that a space with a basis is weak Hilbert whenever there is a $C > 0$ such that every sequence of n vectors that is disjointly supported after n is C-equivalent to the unit vector basis of ℓ_2^n. This property has been called *strongly asymptotic* ℓ_2 and has been studied by several authors in more general settings [5,6,12]. The next proposition shows that $\mathfrak{X}_{wh}$ has this property.

Proposition 4. *The basis $(e_i)_{i\in\mathbb{N}}$ of $\mathfrak{X}_{wh}$ is strongly asymptotic ℓ_2. In particular, for every sequence of disjointly supported vectors $(x_k)_{k=1}^{d}$ such that $d \leqslant \operatorname{supp} x_i$ for $i \in \{1,\dots,d\}$,*

$$\frac{1}{m_2}\left(\sum_{k=1}^{d}\|x_k\|^2\right)^{\frac{1}{2}} \leqslant \left\|\sum_{k=1}^{d} x_k\right\| \leqslant \left(\sum_{k=1}^{d}\|x_k\|^2\right)^{\frac{1}{2}}.$$

The lower inequality follows from Eq. (1) for $j = 1$. The proof of the upper inequality relies on induction on the height of the *tree analysis* of a functional. This notion is central in constructions related to saturation methods (see, for example, [3]) and it essentially describes the "history" of the functional, namely how it is built through the application of the ℓ_2-operations. The key point in this proof is that the coding function is chosen with respect to the sets 'E_i' and not the specific functionals 'f_i.'

As in all constructions of this type, there are several technical obstacles to overcome in order to verify that the space of operators on $\mathfrak{X}_{wh}$ has the desired properties. In the present case, however, the need to evaluate norms with ℓ_2 structure and the fact that the norming functionals are built from others with disjoint and not successive supports requires new techniques for calculating norms. The methods developed for evaluating norms in this modified mixed Tsirelson setting may be of independent interest. It is also worth mentioning that in this context the standard and powerful tool for reducing the complexity of calculations (i.e. the *basic inequality*) is not, and perhaps cannot, be used.

We will state the main proposition required to verify that the operators on block subspaces have the desired decomposition. To do so we recall a few more definitions.

The crucial notion needed is that of a *rapidly increasing sequence* (RIS) of vectors. In our case, the vectors that form a RIS are seminormalized ℓ_2-(ε, n) averages for $\varepsilon > 0$ and $n \in \mathbb{N}$. These averages have been defined previously in the papers [2,4]. Each RIS $(x_n)_n$ is associated with a constant C and an increasing sequence (j_k), where j_{k+1} is chosen inductively with respect to the support of x_k. It is routine to show that every block subspace contains a RIS. Another element we need is the notion of a *diagonal free* operator which is given in the following:

Definition 5. Let X be a Banach space with a Schauder basis $(e_n)_n$ and $T : X \to X$ be a bounded linear operator. T is called diagonal free if $e_n^*(Te_n) = 0$, for all $n \in \mathbb{N}$.

The final step in our work is to show that every diagonal free operator $T \in \mathcal{L}(Y)$, where Y is a block subspace of $\mathfrak{X}_{wh}$ is strictly singular. This actually follows from the fact that for every RIS $(x_n)_n$ in Y, $\lim_n Tx_n = 0$.

In order to show that the diagonal free operators converge in norm to 0 on RIS's, we use the following.

Given a $(C, (i_s))$-RIS (y_n) one can inductively build a further block sequence (x_n) that satisfies the following:

(a) For $j \in \mathbb{N}$ with $j + 1 < j_1 \in N_1$ there is a sequence $(E_i)_{i=1}^{\infty}$ such that $(E_i, 2j_i)_{i=1}^{\infty}$ is σ-special with $(\bigcup_{i\in\mathbb{N}} E_i) \cap (\bigcup_{i\in\mathbb{N}} \operatorname{supp} x_i) = \emptyset$.
(b) The sequence (x_k) is seminormalized and for each $k \in \mathbb{N}$, x_k is a ℓ_2-$(1/m_{2j_k}^3, n_{2j_k})$ average.

Such a sequence (x_k) is called a $(0, C, 2j+1)$ dependent sequence with respect to $(E_i, 2j_i)_{i=1}^{\infty}$.

The critical and most technically demanding inequality that is used for proving the decomposition property of operators on block subspaces and its asymmetrical structure is the following:

Proposition 6. *Let $j \in \mathbb{N}$, $C > 0$, and $(x_k)_{k=1}^{d}$ be a $(0, C, 2j+1)$ dependent sequence and let $\sum_{k=1}^{d} b_k x_k$ be a $(1/m_{2j+2}^2, n_{2j+1})$ average. Then,*

$$\left\| \sum_{k=1}^{d} b_k x_k \right\| \leqslant \frac{C}{m_{2j+1}^2}.$$

Granting the above, one can show that a diagonal free operator goes to zero along an RIS. The proof essentially reduces (using a technical counting argument from [7]) to showing that for every RIS $(x_n)_n$ and every partition C_n, B_n of supp x_n, $\lim_n C_n T B_n x_n = 0$. This argument relies heavily on Proposition 6; we give an outline of the proof below.

Proposition 7. *Let Y be a block subspace of $\mathfrak{X}_{wh}$ generated by $(y_n)_n$ and $T : Y \to Y$ be a diagonal free operator with respect to $(y_n)_n$. Let also $(x_n)_n$ be a RIS in Y, then $Tx_n \to 0$.*

Proof. Suppose, toward a contradiction, that the conclusion fails. Then, from the preceding discussion by passing to a subsequence if necessary, there exists $\varepsilon > 0$ such that $\|C_n T B_n x_n\| > \varepsilon$ for every $n \in \mathbb{N}$. It is easy to see that $(B_n x_n)$ is a $(C, (2i_n))$-RIS. For each $n \in \mathbb{N}$ let $f_n \in D_{wh}$ such that $f_n(C_n T B_n x_n) > \varepsilon$ and supp $f_n \subset C_n$. Choose $j \in \mathbb{N}$ such that $\frac{1}{m_{2j+1}} < \frac{\varepsilon}{\|T\|C}$. Next we construct sequences $(z_k)_{k=1}^{\infty}$, $(g_k)_{k=1}^{\infty}$ and $(E_i, 2j_i)_{i=1}^{\infty}$ such that,

(1) $(z_k)_{k=1}^{\infty}$ is a block subsequence of (x_n).
(2) $(z_k)_{k=1}^{\infty}$ is a $(0, C, 2j+1)$ dependent sequence with respect to $(E_i, 2j_i)_{i=1}^{\infty}$.
(3) $(g_k)_{k=1}^{\infty} \subset D_{wh}$ is σ-special with respect to $(E_i, 2j_i)_{i=1}^{\infty}$.
(4) $g_k(Tz_k) \geqslant \varepsilon$ for all $k \in \mathbb{N}$.

Given this, we arrive at a contraction in the following way: Find $d \in \mathbb{N}$ such that $(\min E_i)_{i=1}^{d}$ is a maximal element of $\mathcal{S}_{n_{2j+1}}$. There is a sequence $(b_k)_{k=1}^{d} \in Ba(\ell_2)$ such that $\sum_{k=1}^{d} b_k z_k$ is a $(1/m_{2j+2}^2, n_{2j+1})$ average. Using the conditions on the sequences and Proposition 6, the contradiction to our choice of j is as follows,

$$\frac{\varepsilon}{m_{2j+1}} < \frac{1}{m_{2j+1}} \sum_{k=1}^{d} b_k g_k \left(\sum_{k=1}^{d} b_k T z_k \right) \leqslant \left\| T \left(\sum_{k=1}^{d} b_k z_k \right) \right\| \leqslant \frac{C\|T\|}{m_{2j+1}^2}. \qquad \square$$

The above proposition yields that every diagonal free operator is strictly singular. This fact combined with the unconditionality of the basis of $\mathfrak{X}_{wh}$ yield the following main result of the Note:

Theorem 8. *Let Y be a block subspace of $\mathfrak{X}_{wh}$ generated by $(y_n)_n$ and $T : Y \to Y$ be a bounded linear operator. Then T can be written as $T = D + S$, where D is a diagonal operator with respect to the basis $(y_n)_n$ of Y and S is a strictly singular operator. Moreover, every block subspace of $\mathfrak{X}_{wh}$ cannot be isomorphic to any of its proper subspaces.*

It is easy to see that every bounded linear operator $T : Y \to Y$ is the sum of a diagonal operator D and a diagonal free operator S and from Proposition 7 we conclude that S is strictly singular. The last part of Theorem 8 follows from the decomposition of T and Fredholm theory [7].

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