C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1273–1277

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Ordinary Differential Equations

Composite asymptotic expansions and turning points of singularly perturbed ordinary differential equations

Développements asymptotiques combinés et points tournants d'équations différentielles singulièrement perturbées

Augustin Fruchard [a], Reinhard Schäfke [b]

[a] *LMIA, EA 3993, Université de haute Alsace, 4, rue des Frères-Lumière, 68093 Mulhouse cedex, France*
[b] *IRMA, UMR 7501, Université de Strasbourg et CNRS, 7, rue René-Descartes, 67084 Strasbourg cedex, France*

A R T I C L E I N F O

Article history:
Received 4 May 2010
Accepted 18 October 2010
Available online 10 November 2010

Presented by Jean-Pierre Ramis

A B S T R A C T

We present a new type of asymptotic expansions for functions of two variables, the coefficients of which contain functions of one of the variables as well as functions of the quotient of these two variables. These combined asymptotic expansions (CAE) are particularly well suited for the description of solutions of singularly perturbed ordinary differential equations in the neighborhood of turning points. The relations with the method of matched asymptotic expansions and with the classical CAE used for boundary layers are described. An application to canard solutions is given.

R É S U M É

On présente une théorie de développements asymptotiques pour des fonctions de deux variables, combinant à la fois des fonctions d'une des variables et des fonctions du quotient de ces deux variables. Ces développements asymptotiques combinés (DAC) sont bien adaptés à la description des solutions d'équations différentielles ordinaires singulièrement perturbées au voisinage de points tournants. Le lien et les différences avec les méthodes de matching et les développements combinés classiques sont décrits. Cette théorie est appliquée à un problème de solutions canard.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

Cette Note présente une sélection de résultats du mémoire [5]. Nous renvoyons le lecteur à ce mémoire pour des preuves détaillées, des définitions supplémentaires et des résultats intermédiaires.

Étant donnés $\alpha < \beta \leqslant \alpha + 2\pi$, $0 < r \leqslant +\infty$ et $\mu \in \mathbb{R}$, le *quasi-secteur* $V(\alpha, \beta, r, \mu)$ est, ou bien l'union du secteur $S(\alpha, \beta, r) = \{x \in \mathbb{C};\ 0 < |x| < r,\ \alpha < \arg x < \beta\}$ et du disque $D(0, \mu) = \{x \in \mathbb{C};\ |x| < \mu\}$ si $\mu > 0$, ou bien le secteur $S(\alpha, \beta, r)$ écorné par le disque $D(0, |\mu|)$ si $\mu < 0$.

E-mail addresses: augustin.fruchard@uha.fr (A. Fruchard), schaefke@unistra.fr (R. Schäfke).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.10.027

Étant donné un quasi-secteur infini $V = V(\alpha, \beta, \infty, \mu)$, $\mathcal{G}(V)$ désigne l'ensemble des fonctions holomorphes et bornées sur V, ayant un développement asymptotique sans terme constant quand $V \ni X \to \infty$. Pour $r_0 > 0$, $\mathcal{H}(r_0)$ désigne l'ensemble des fonctions holomorphes et bornées sur le disque $D(0, r_0)$.

Définition 0.1. On dira qu'une fonction $y = y(x, \eta)$, définie et holomorphe pour $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ et $x \in V_1(\eta) = V(\alpha_1, \beta_1, r_0, \mu|\eta|)$, a un *développement asymptotique combiné* (DAC en abrégé) s'il existe $\alpha \leqslant \alpha_1 - \beta_2$, $\beta \geqslant \beta_1 - \alpha_2$ et deux suites de fonctions $(a_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, $a_n \in \mathcal{H}(r_0)$, $g_n \in \mathcal{G}(V)$ avec $V = V(\alpha, \beta, r, \mu)$, vérifiant (1). On note dans ce cas $\widehat{y} = \sum_{n\geqslant 0}(a_n(x) + g_n(\frac{x}{\eta}))\eta^n$ et $y(x, \eta) \sim \widehat{y}$ quand $S_2 \ni \eta \to 0$ et $x \in V_1(\eta)$. Étant donné $p \in \mathbb{N}^*$, on dira que *y admet* $\widehat{y}$ *pour* DAC *Gevrey d'ordre* $\frac{1}{p}$, et on écrira $y(x, \eta) \sim_{1/p} \widehat{y}$ quand $S_2 \ni \eta \to 0$ et $x \in V_1(\eta)$, s'il existe $C, L_1, L_2 > 0$ tels que (1) est satisfait avec $K_N = CL_1^N \Gamma(\frac{N}{p} + 1)$ et si de plus (2) est satisfait.

Sous des conditions appropriées, ces DAC sont compatibles avec les opérations usuelles d'addition, multiplication, dérivation, intégration, division et composition par des fonctions analytiques. Ils sont aussi étroitement liés à la méthode de recollement de développements intérieurs et extérieurs. Précisément, si une fonction $y = y(x, \eta)$ admet un DAC, resp. DAC Gevrey-$\frac{1}{p}$, $\sum_{n\geqslant 0}(a_n(x) + g_n(\frac{x}{\eta}))\eta^n$ quand $S_2 \ni \eta \to 0$ et $x \in V_1(\eta)$, alors pour $x \in S(\alpha_1, \beta_1, x_0)$ fixé on a le *développement extérieur* (4). De plus, pour tout $\delta > 0$, ce développement est uniforme pour les $x \in S(\alpha_1, \beta_1, x_0)$ tels que $|x| > \delta$. De même, si $X \in V$ et α_3, β_3, η_3 sont tels que $\eta \in S(\alpha_3, \beta_3, \eta_3)$ implique $\eta \in S_2$ et $\eta X \in V_1(\eta)$, alors on a le *développement intérieur* (5), uniforme sur les parties compactes de V satisfaisant la condition précédente. Enfin, ces DAC sont prolongeables : si une fonction a un DAC pour x dans un quasi-secteur et si le développement extérieur, resp. intérieur, existe dans un quasi-secteur plus grand, alors le DAC est valide dans le grand quasi-secteur. Le caractère Gevrey est aussi conservé.

On considère une équation différentielle singulièrement perturbée de la forme (6) avec f analytique dans le disque $D(0, r_0)$ et vérifiant $f(x) = px^{p-1} + \mathcal{O}(x^p)$ quand $x \to 0$ où $p \in \mathbb{N}$, $p \geqslant 2$, h analytique dans $D(0, r_0) \times D(0, \varepsilon_0)$, P analytique dans $D(0, r_0) \times D(0, r_2) \times D(0, \varepsilon_0)$, $P(x, 0, 0) \equiv 0$. Ainsi (6) a un point tournant à l'origine d'ordre $p - 1$; le petit paramètre ε est relié à η par $\varepsilon = \eta^p$. Les solutions de (6) seront considérées comme des fonctions de x et η. Soit F la primitive de f s'annulant en $x = 0$ et R la partie réelle de F. Le *relief* associé à (6) est le graphe de R. Il est composé d'une succession de p montagnes $\mathcal{M}_j$ (où $R > 0$) et p vallées $\mathcal{V}_j$ (où $R < 0$) séparées par la séparatrice (où $R = 0$) du col $x = 0$. On numérote ces montagnes et vallées de telle manière que $\mathcal{M}_0$ contienne une partie de $\mathbb{R}^+$ et qu'elles alternent $\mathcal{M}_j, \mathcal{V}_j, \mathcal{M}_{j+1}$ mod p. Ainsi, $\mathcal{M}_j$ est tangent à l'origine au secteur $S(\frac{2j\pi}{p} - \frac{\pi}{2p}, \frac{2j\pi}{p} + \frac{\pi}{2p}, \infty)$ et $\mathcal{V}_j$ à $S(\frac{2j\pi}{p} + \frac{\pi}{2p}, \frac{2j\pi}{p} + \frac{3\pi}{2p}, \infty)$.

On sait qu'à chaque montagne est associée une solution d'ordre $\mathcal{O}(\varepsilon)$ et qu'elle a un développement asymptotique en puissances de $\varepsilon = \eta^p$. Cependant, cette description n'est pas valide proche du point tournant. L'énoncé suivant fournit une description de ces solutions valide aussi près du point tournant.

Théorème 0.2. *Étant donnée l'Éq.* (6) *avec les hypothèses d'analyticité, on suppose de plus qu'il existe* $r \in \mathbb{N}^*$ *tel que* $h(x, 0) = \mathcal{O}(x^{r-1})$ *lorsque* $x \to 0$ *et* $P(x, y, 0) = \sum_{k\geqslant 0, l\geqslant 1, k+rl\geqslant p-1} p_{kl}x^k y^l$. *Soit* $\alpha, \beta \in \mathbb{R}$, $\delta > 0$ *et* $r_1 \in\,]0, r_0(\cos(2p\delta))^{1/p}[$ *tels que* $S(\alpha, \beta, r_1) \subset \mathcal{V}_{j-1} \cup \mathcal{M}_j \cup \mathcal{V}_j$.

Alors il existe $\mu \in \mathbb{R}$, $\eta_1 > 0$ *et une solution y définie pour* $\eta \in S_1 := S(-\delta, \delta, \eta_1)$ *et* $x \in V(\eta) := V(\alpha + 3\delta, \beta - 3\delta, r_1 - \delta, \mu|\eta|)$, *vérifiant* $y(x, \eta) = \mathcal{O}(\eta^r)$. *De plus y a un* DAC *Gevrey d'ordre* $\frac{1}{p}$ *quand* $S_1 \ni \eta \to 0$ *et* $x \in V(\eta)$.

L'*équation intérieure* (7) permet de préciser des valeurs de μ possibles.

Théorème 0.3. *Soit* Y_0 *l'unique solution de* (7) *telle que* $Y_0(X) \sim -\frac{c}{p}X^{r-p}$ *quand* $|X| \to \infty$ *dans un quasi-secteur* $V = V(\alpha, \beta, \infty, \mu)$, $\alpha < 0 < \beta$, $\mu < 0$ *assez grand. Si* Y_0 *se prolonge dans un voisinage de l'adhérence de* $V(\alpha, \beta, \infty, \widetilde{\mu})$ *pour un certain* $\widetilde{\mu} > \mu$, *alors pour* η_1 *assez petit y se prolonge et a un* DAC *quand* $S_1 \ni \eta \to 0$ *et* $x \in \widetilde{V}(\eta) = V(\alpha + 3\delta, \beta - 3\delta, r_0, \widetilde{\mu}|\eta|)$.

En particulier, si $p_{kl} = 0$ pour $k + rl = p - 1$, alors (7) est linéaire et tout $\widetilde{\mu} \in \mathbb{R}$ convient.

Dans [5], trois applications de nos DAC sont détaillées. La première est une condition formelle nécessaire et suffisante pour l'existence de canards. Si f est à valeurs réelles sur un intervalle $[a, b]$ avec $a < 0 < b$ et si $xf(x) > 0$ pour $x \neq 0$ réel, l'axe réel correspond à deux montagnes ayant chacune une solution $y^{\pm}$ munie d'un DAC Gevrey-$\frac{1}{p}$. On montre alors que l'équation a des canards si et seulement si les *coefficients rapides* g_n^+ et g_n^- se prolongent analytiquement sur $\mathbb{R}$ et coïncident pour tout $n \in \mathbb{N}$. Les deux autres applications concernent les canards non lisses et la résonance d'Ackerberg–O'Malley.

1. Introduction

This Note presents a selection of results of the memoir [5]. We refer the reader to this memoir for exhaustive proofs, additional definitions and intermediate results.

Consider a singularly perturbed ODE of the form (6) with f, h, P analytic on suitable domains. We say that x^* is a *regular point* if $f(x^*) \neq 0$, and a *turning point* otherwise.

If x^* is regular and $v : D(0, \varepsilon_0) \to \mathbb{C}$ is analytic with $v(0)$ small enough, it is known that the solution y of (6) with initial condition $y(x^*, \varepsilon) = v(\varepsilon)$ has a *composite asymptotic expansion* (CAE for short) of the form $y(x, \varepsilon) \sim \sum_{n\geqslant 0}(y_n(x) +$

$z_n(\frac{x-x^*}{\varepsilon}))\varepsilon^n$, where $y_0 \equiv 0$, $\sum_{n\geqslant 1} y_n(x)\varepsilon^n$ is the unique formal solution of (6) and the functions $z_n = z_n(X)$ are exponentially decreasing functions as $\mathrm{Re}(X)$ tends to $-\infty$; see e.g. the work of Vasil'eva and Butuzov [7] or the presentation and bibliography in [1].

At a turning point, even if f is analytic, each function y_n usually has a pole of order np, and the classical method of CAE no longer applies. Another classical method adapted to turning points is that of *matching* of so-called *inner* and *outer expansions*, see e.g. [3,8]. Here we present new CAE, well-adapted to turning points and *canard* (French duck) problems. An alternative approach of CAE at turning points is described in Forget's thesis [4].

In this presentation, our CAE are applied only to scalar differential equations, but they may serve to various functional equations such as differential systems, difference or q-difference equations and systems.

2. Combined expansions

Given $\alpha < \beta \leqslant \alpha + 2\pi$, $0 < r \leqslant +\infty$ and $\mu \in \mathbb{R}$, the *quasi-sector* $V(\alpha, \beta, r, \mu)$ is either the union of the sector $S(\alpha, \beta, r) = \{x \in \mathbb{C};\ 0 < |x| < r,\ \alpha < \arg x < \beta\}$ and the disk $D(0, \mu) = \{x \in \mathbb{C};\ 0 < |x| < \mu\}$ if $\mu > 0$, or the part of $S(\alpha, \beta, r)$ outside $D(0, |\mu|)$ if $\mu < 0$.

Given an infinite quasi-sector $V = V(\alpha, \beta, \infty, \mu)$, $\mathcal{G}(V)$ denotes the set of functions holomorphic and bounded on V that have an asymptotic expansion without constant term as $V \ni X \to \infty$.

Given $r_0 > 0$, $\mathcal{H}(r_0)$ denotes the set of functions holomorphic and bounded on the disk $D(0, r_0)$.

Definition 2.1. We say that a function $y = y(x, \eta)$, defined and holomorphic for $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in V_1(\eta) = V(\alpha_1, \beta_1, r_0, \mu|\eta|)$, has a *composite asymptotic expansion* if there exist $\alpha \leqslant \alpha_1 - \beta_2$, $\beta \geqslant \beta_1 - \alpha_2$ and two sequences $(a_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, $a_n \in \mathcal{H}(r_0)$, $g_n \in \mathcal{G}(V)$, $V = V(\alpha, \beta, r, \mu)$, such that

$$\forall N \in \mathbb{N}^*\ \exists K_N > 0\ \forall \eta \in S_2\ \forall x \in V_1(\eta), \qquad \left| y(x, \eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n\left(\frac{x}{\eta}\right) \right) \eta^n \right| \leqslant K_N |\eta|^N. \tag{1}$$

In that case, we note $\widehat{y} = \sum_{n\geqslant 0}(a_n(x) + g_n(\frac{x}{\eta}))\eta^n$ and $y(x, \eta) \sim \widehat{y}$ as $S_2 \ni \eta \to 0$ and $x \in V_1(\eta)$. The series $\sum_{n\geqslant 0} a_n(x)\eta^n$ is the *slow part* of the CAE and $\sum_{n\geqslant 0} g_n(\frac{x}{\eta})\eta^n$ is the *fast part*.

Given $p \in \mathbb{N}^*$, we say that y has $\widehat{y}$ as a *Gevrey* CAE *of order* $\frac{1}{p}$, and we write $y(x, \eta) \sim_{1/p} \widehat{y}$ as $S_2 \ni \eta \to 0$ and $x \in V_1(\eta)$, if there exist $C, L_1, L_2 > 0$ such that (1) is satisfied with $K_N = C L_1^N \Gamma(\frac{N}{p} + 1)$ and if

$$\forall n \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall X \in V \setminus \{0\}, \qquad \left| g_n(X) - \sum_{m=0}^{M-1} g_{nm} X^{-m} \right| \leqslant C L_1^n L_2^M \Gamma\left(\frac{n+M}{p} + 1\right) |X|^{-M}. \tag{2}$$

Under appropriate conditions, these CAE are compatible with the usual operations of addition, multiplication, differentiation, integration, division and composition with analytic functions.

For the multiplication of two CAE, products of the form $a(x)g(\frac{x}{\eta})$ have to be developed. This is done using the shift operators $\mathbf{S} : \mathcal{H}(x_0) \to \mathcal{H}(x_0)$ and $\mathbf{T} : \mathcal{G}(V) \to \mathcal{G}(V)$ defined by $\mathbf{S}a(x) = \frac{1}{x}(a(x) - a(0))$ and $\mathbf{T}g(X) = Xg(X) - g_1$. With the notation $a(x) = \sum_{n\geqslant 0} a_n x^n$ and $g(X) \sim \sum_{n\geqslant 1} g_n X^{-n}$, observing that $a(x)g(\frac{x}{\eta}) = a_0\, g(\frac{x}{\eta}) + g_1 \mathbf{S}a(x)\eta + \mathbf{S}a(x)\mathbf{T}g(\frac{x}{\eta})\eta$, one obtains (with $g_0 = 0$)

$$a(x)g\left(\frac{x}{\eta}\right) \sim \sum_{\nu\geqslant 0} \left(g_\nu \left(\mathbf{S}^\nu a\right)(x) + a_\nu \left(\mathbf{T}^\nu g\right)\left(\frac{x}{\eta}\right) \right) \eta^\nu. \tag{3}$$

The proof of the compatibility with differentiation uses Cauchy's formula and therefore the domains have to be reduced. For the integration of CAE, it is unavoidable to take care of the residues $\widehat{R}(\eta) = \sum_{n\geqslant 0} g_{n1}\eta^n$ of the functions g_n. The question whether the multiplicative inverse of a function having a CAE has a CAE or not is slightly more complicated and is discussed in detail in [5]. There, necessary and sufficient conditions involving the *inner* or *outer* expansions (presented below) are given.

Our CAE are closely related to the method of matched asymptotic expansions. The result is as follows.

Proposition 2.2. *Given $a_n \in \mathcal{H}(r_0)$, $g_n \in \mathcal{G}(V)$, with expansions $a_n(x) = \sum_{m=0}^{\infty} a_{nm} x^m$ and $g_n(X) \sim \sum_{m>0} g_{nm} X^{-m}$, assume that $y(x, \eta)\sim$, resp. $\sim_{1/p} \sum_{n\geqslant 0}(a_n(x) + g_n(\frac{x}{\eta}))\eta^n$ as $S_2 \ni \eta \to 0$, $x \in V_1(\eta)$.*

Then for fixed $x \in S(\alpha_1, \beta_1, x_0)$ one has the so-called outer expansion

$$y(x, \eta)\sim, \quad \textit{resp.}\ \sim_{1/p} \sum_{n\geqslant 0} c_n(x)\eta^n \quad \textit{as } S_2 \ni \eta \to 0,\ \textit{with } c_n(x) = a_n(x) + \sum_{l=0}^{n-1} g_{l,n-l} x^{l-n}. \tag{4}$$

Moreover, for all $\delta > 0$, this expansion is uniform with respect to $x \in S(\alpha_1, \beta_1, x_0)$ such that $|x| > \delta$.

Similarly, if $X \in V$ and $\alpha_3, \beta_3, \eta_3$ are such that $\eta \in S_3 = S(\alpha_3, \beta_3, \eta_3)$ implies $\eta \in S_2$ and $\eta X \in V_1(\eta)$, then one has the inner expansion

$$y(\eta X, \eta) \sim, \quad \textit{resp.}\ \sim_{1/p} \sum_{n=0}^{\infty} h_n(X)\eta^n \quad \textit{as } S_3 \ni \eta \to 0 \textit{ with } h_n(X) = \sum_{l=0}^{n} a_{n-l,l} X^l + g_n(X). \tag{5}$$

This expansion is uniform with respect to X in compact subsets of V satisfying the former condition.

One can show that, for all $\kappa \in]0,1[$, the first expansion is valid uniformly for $|x| > |\eta|^\kappa$, whereas the second one is valid uniformly for $|X| < |\eta|^{-\kappa}$; Gevrey estimates would also depend on κ, though.

In some applications, CAE may therefore be useful to legitimate the method of matched asymptotic expansions. However, it is often preferable to have uniform approximations on the whole domain of study.

Conversely, these inner and outer expansions are useful to compute a CAE, if we already know that it exists. For x far from 0, one computes the outer expansion $y(x,\eta) \sim \sum_{n \geqslant 0} c_n(x)\eta^n$, then throws away negative powers of x, obtaining the slow terms $a_n(x)$. Then one computes the inner expansion $y(\eta X, \eta) \sim \sum_{n \geqslant 0} h_n(X)\eta^n$ and rejects the polynomial part, yielding the fast terms $g_n(X)$. In practice the computation of inner and outer expansions often leads to recursive equations for the coefficients. This provides a_n, g_n without using the cumbersome formula (3) of multiplication of CAE.

There exist also results concerning the extension of CAE, see [5]. Roughly speaking, these results state that, if a function has a CAE for x in some quasi-sector, and if the outer, resp. inner, expansion remains valid in a larger quasi-sector, then also the CAE remains valid in the large quasi-sector.

3. Singular perturbation

Consider a singularly perturbed ODE of the form

$$\varepsilon y' = f(x)y + \varepsilon h(x,\varepsilon) + yP(x,y,\varepsilon) \tag{6}$$

with f analytic in the disk $D(0,r_0)$ and satisfying $f(x) = px^{p-1} + \mathcal{O}(x^p)$ as $x \to 0$ where $p \in \mathbb{N}, p \geqslant 2$, h analytic in $D(0,r_0) \times D(0,\varepsilon_0)$, P analytic in $D(0,r_0) \times D(0,r_2) \times D(0,\varepsilon_0)$, $P(x,0,0) \equiv 0$. Eq. (6) has a turning point of order $p-1$ at the origin; the small parameter ε is linked to η by $\varepsilon = \eta^p$. Although the equation is expressed in terms of ε, for convenience we consider solutions of (6) as functions of x and η. Let F denote the antiderivative of f that vanishes at $x = 0$ and let R denote its real part. The *landscape* associated to (6) is the graph of R. It is made of a succession of p mountains $\mathcal{M}_j$ (where $R > 0$) and p valleys $\mathcal{V}_j$ (where $R < 0$) separated by the separatrix (where $R = 0$) of the saddle point $x = 0$. These mountains and valleys are numbered such that $\mathcal{M}_0$ contains a part of $\mathbb{R}^+$ and that they alternate $\mathcal{M}_j, \mathcal{V}_j, \mathcal{M}_{j+1}$ mod p. In this manner, $\mathcal{M}_j$ is tangent at the origin to the sector $S(\frac{2j\pi}{p} - \frac{\pi}{2p}, \frac{2j\pi}{p} + \frac{\pi}{2p}, \infty)$ and $\mathcal{V}_j$ to $S(\frac{2j\pi}{p} + \frac{\pi}{2p}, \frac{2j\pi}{p} + \frac{3\pi}{2p}, \infty)$.

It is known that, on each mountain, solutions of (6) exist which are $\mathcal{O}(\varepsilon)$ and that these solutions have asymptotic expansions in powers of $\varepsilon = \eta^p$. This description, however, is valid not near the turning point. In the following statement we provide a description of such solutions close to the turning point:

Theorem 3.1. *Consider Eq. (6) with the analyticity assumptions. Assume moreover that there exists $r \in \mathbb{N}^*$ such that $h(x,0) = \mathcal{O}(x^{r-1})$ as $x \to 0$ and*

$$P(x,y,0) = \sum_{k \geqslant 0,\, l \geqslant 1,\, k+rl \geqslant p-1} p_{kl} x^k y^l.$$

Let $\alpha, \beta \in \mathbb{R}$, $\delta > 0$ and $r_1 \in]0, r_0(\cos(2p\delta))^{1/p}[$ be such that $S(\alpha,\beta,r_1) \subset \mathcal{V}_{j-1} \cup \mathcal{M}_j \cup \mathcal{V}_j$.

Then there exist $\mu \in \mathbb{R}, \eta_1 > 0$ and a solution y defined for $\eta \in S_1 := S(-\delta,\delta,\eta_1)$ and $x \in V(\eta) := V(\alpha + 3\delta, \beta - 3\delta, r_1 - \delta, \mu|\eta|)$, satisfying $y(x,\eta) = \mathcal{O}(\eta^r)$. Moreover, y has a CAE Gevrey of order $\frac{1}{p}$ as $S_1 \ni \eta \to 0$ and $x \in V(\eta)$.

The proof uses a Ramis–Sibuya type theorem. The statement is still valid if g and P are analytic and Gevrey-1 in ε only in a sector instead of the disk $D(0,\varepsilon_0)$. The example $\varepsilon y' = 4x^3 y - 4\varepsilon - xy^2$ shows that the assumption on $P(x,y,0)$ is crucial. Information about μ can be obtained from the *interior equation*

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = pX^{p-1}Y + cX^{r-1} + Y^2 Q(X,Y). \tag{7}$$

It is obtained replacing $x = \eta X$, $y = \eta^r Y$ in (6) and letting $\eta \to 0$. This yields $c = \lim_{x\to 0} x^{1-r} h(x,0)$ and $Q(X,Y) = \sum_{k+rl=p-1} p_{kl} X^k Y^{l-1}$. The classical theory of irregular singular points shows that (7) has a unique solution $Y_0(X) \sim -\frac{c}{p} X^{r-p}$ as $|X| \to \infty$ in a quasi-sector $V = V(-\frac{3\pi}{2p} + \gamma, \frac{3\pi}{2p} - \gamma, \infty, \mu)$ for arbitrary $\gamma > 0$ and for some $\mu < 0$, $|\mu|$ large enough. Using the general result concerning extension of CAE, the following statement can be deduced:

Theorem 3.2. *With the notation and assumptions of Theorem* 3.1*, suppose that $\alpha, \beta \in \mathbb{R}$ and $\widetilde{\mu} > \mu$ are such that Y_0 can be analytically continued in $V(\alpha, \beta, \infty, \widetilde{\mu})$. Then, for η_1 small enough, y can be analytically continued and has a* CAE *as $S_1 \ni \eta \to 0$ and $x \in \widetilde{V}(\eta) = V(\alpha + 3\delta, \beta - 3\delta, r_0, \widetilde{\mu}|\eta|)$.*

A case of particular interest occurs when $p_{kl} = 0$ for $k + rl = p - 1$, i.e. $Q \equiv 0$. Then Eq. (7) is linear, hence for all $\widetilde{\mu} \in \mathbb{R}$ there is $\eta_1 > 0$ such that y is defined and has a CAE as $S_1 \ni \eta \to 0$, $x \in V(\eta)$.

As an application of Theorem 3.1, consider f analytic in a neighborhood of some real interval $[a, b]$ with $a < 0 < b$, such that f is real valued on the real axis and $xf(x) > 0$ for real $x \neq 0$. Then p is even and the real axis is the ridge of two mountains, $\mathcal{M}^-$ containing $[a, 0[$ and $\mathcal{M}^+$ containing $]0, b]$. On each mountain there is a solution, y^- on $\mathcal{M}^-$ and y^+ on $\mathcal{M}^+$, which has a Gevrey-$\frac{1}{p}$ CAE $y^{\pm}(x, \eta) \sim_{1/p} \sum_{n \geqslant 0} (a_n(x) + g_n^{\pm}(\frac{x}{\eta}))\eta^n$ as $S_1 \ni \eta \to 0$ and $x \in V(\pi - \delta, \pi + \delta, |a|, \mu|\eta|)$ for y^-, resp. $x \in V(-\delta, \delta, b, \mu|\eta|)$ for y^+, if $\delta, \eta_0 > 0$ are small enough and $\mu < 0$ is large enough. In general, there is no solution of (6) defined and close to 0 in a neighborhood of $x = 0$ independent of ε. If such a solution exists, it is called a *local canard* solution. This situation is exceptional for a single equation but often appears in a one-parameter family of equations, see e.g. Chapter 10 of [2]. A *global canard* solution, if it exists, is a solution defined and close to 0 for $\eta \in S_1$ and $x \in [a, b]$.

Theorem 3.3. *The following assertions are equivalent*:

a. *There exists a local canard solution.*
b. *There exists a global canard solution.*
c. *For each $n \in \mathbb{N}$, the functions g_n^+ and g_n^- can be analytically continued on $\mathbb{R}$ and coincide.*

The equivalence of a. and b. is already in [6]. Here we add a condition based on formal solutions of (6). In [5], the CAE are also applied to non-smooth canard solutions and to the Ackerberg–O'Malley resonance.

References

[1] E. Benoît, A. El Hamidi, A. Fruchard, On combined asymptotic expansions in singular perturbations, Electron. J. Diff. Eqns. 2002 (51) (2002) 1–27.
[2] F. Diener, M. Diener, Nonstandard Analysis in Practice, Universitext, Springer, 1995.
[3] W. Eckhaus, Asymptotic Analysis of Singular Perturbations, Studies in Mathematics and Its Applications, vol. 9, North-Holland, 1979.
[4] T. Forget, Points tournants dégénérés, Thèse de Doctorat, Université de La Rochelle, 2007.
[5] A. Fruchard, R. Schäfke, Développements asymptotiques combinés et points tournants d'équations différentielles singulièrement perturbées, preprint HAL: hal-00461542 and arXiv:1004.5254.
[6] P. De Maesschalck, On maximum bifurcation delay in real planar singularly perturbed vector fields, Nonlinear Analysis 68 (3) (2008) 547–576.
[7] A.B. Vasil'eva, V.F. Butuzov, Asymptotic Expansions of the Solutions of Singularly Perturbed Equations, Izdat. "Nauka", Moscow, 1973 (in Russian).
[8] W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Interscience, New York, 1965.