



Partial Differential Equations/Mathematical Problems in Mechanics

## Local in time strong solvability of the non-steady Navier–Stokes equations with Navier’s boundary condition and the question of the inviscid limit

*Solutions fortes vérifiant des conditions aux limites de Navier pour les équations de Navier–Stokes non stationnaires, et la question de leur limite inviscide*

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### ABSTRACT

In this Note, we prove the existence of strong solutions to the Navier–Stokes equations for incompressible viscous fluids in a general regular bounded domain of  $\mathbb{R}^3$  on a “short” time interval  $(0, T_0)$ , independent of the viscosity and of the friction between the fluid and the boundary. The solutions to the Navier–Stokes problem satisfy the inhomogeneous Navier’s boundary condition and they reveal a remarkable structure of approximation of the solution to the Euler problem, which enables us to solve completely the question of the inviscid limit of the family of obtained solutions on the time interval  $(0, T_0)$ .

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### R É S U M É

Dans cette Note, nous démontrons l’existence locale en temps de solutions fortes pour les équations de Navier–Stokes descriptives de fluides visqueux incompressibles, dans un domaine borné de  $\mathbb{R}^3$ , général et suffisamment régulier, avec des conditions aux limites non homogènes de Navier bien choisies. Ces solutions sont construites avec la même structure remarquable d’approximation de la solution du problème d’Euler que celles obtenues avec des conditions d’imperméabilité généralisées ou des conditions de type celles de Navier : structure permettant de traiter complètement la question de la limite inviscide.

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### Version française abrégée

Soit  $Q_T = \Omega \times (0, T)$  où  $T > 0$  et  $\Omega$  un domaine borné de  $\mathbb{R}^3$  de frontière  $\partial\Omega$  imperméable et régulière de classe  $C^{3,1}$ . Dans  $Q_T$ , l’écoulement d’un fluide Newtonien visqueux incompressible est usuellement décrit par le système des équations de Navier–Stokes

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0, \quad (i)$$

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avec  $\nu$  le coefficient de viscosité cinématique :  $\mathbf{u}$  est la vitesse,  $p$  est la pression associée, et  $(\nabla \mathbf{u})_s = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$  notera le gradient de vitesse symétrisé décrivant le tenseur de cisaillement  $\mathbb{T}_d^v(\mathbf{u}) = 2\nu(\nabla \mathbf{u})_s$ . Et naturellement l'imperméabilité de  $\partial\Omega$  s'exprime par la condition aux limites  $\mathbf{u} \cdot \mathbf{n} = 0$  sur  $\partial\Omega \times (0, T)$ , soit (ii). Lorsque  $\nu = 0$ , avec une donnée initiale  $\mathbf{u}(\cdot, 0) = \mathbf{u}^*$  dans  $\Omega$ , soit (iii), le problème (i) $_{\nu=0}$ , (ii), (iii) est bien posé, il s'agit du problème d'Euler, et l'existence locale en temps d'une unique solution forte  $\mathbf{u}^0$  est bien connue : « si la condition initiale est donnée avec  $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}_\sigma^2(\Omega)$ ,  $\operatorname{div} \mathbf{u}^* = 0$  dans  $\Omega$  et  $\mathbf{u}^* \cdot \mathbf{n} = 0$  sur  $\partial\Omega$ , pourvu que  $\mathbf{f} \in L^1(0, T; \mathbf{W}^{4,2}(\Omega))$ , il existe  $\mathbf{u}^0$  unique solution de (1) $_{\nu=0}$ , (ii), (iii) vérifiant  $\mathbf{u}^0 \in L^\infty(0, T_0; \mathbf{W}^{4,2}(\Omega))$  et  $\partial_t \mathbf{u}^0 \in L^1(0, T_0; \mathbf{W}^{4,2}(\Omega)) + L^\infty(0, T_0; \mathbf{W}^{3/2,2}(\Omega))$  » [9].

Lorsque  $\nu \neq 0$ , nous considérons le problème aux limites  $(\mathbf{P}_{\nu,\gamma})$  défini par les équations (i), (ii), (iii) auxquelles nous ajoutons sur  $\partial\Omega \times (0, T)$  les conditions tangentielles

$$[\mathbb{T}_d^v(\mathbf{u}) \cdot \mathbf{n}]_\tau + \gamma \mathbf{u} = \mathbf{a}^{\nu,\gamma}. \quad (\text{iv})$$

Proposées par Navier dans le cas homogène  $\mathbf{a}^{\nu,\gamma} = 0$ , les conditions aux limites (iv) complètent la condition de flux nul (ii) et expriment le glissement avec frottement à la paroi  $\partial\Omega$ ,  $\gamma$  étant le coefficient de frottement. Nous choisissons ici leur analogue non homogène avec la donnée d'une fonction  $\mathbf{a}^{\nu,\gamma}$  la plus souhaitable et la plus « naturelle » possible pour prédire, avec la structure

$$\mathbf{u} = \mathbf{u}^0 + \nu \mathbf{v}^0 + \nu \mathbf{w}^{\nu,\gamma}, \quad (\text{v})$$

une solution  $\nu$ -continue au problème  $(\mathbf{P}_{\nu,\gamma})$  : Posons

$$\mathbf{a}^{\nu,\gamma} := [\mathbb{T}_d^v(\mathbf{u}^0 + \nu \mathbf{v}^0) \cdot \mathbf{n}]_\tau + \gamma(\mathbf{u}^0 + \nu \mathbf{v}^0)_\tau \quad (\text{vi})$$

où  $\mathbf{u}^0$  est la solution du problème d'Euler (i) $_{\nu=0}$ , (ii), (iii), et où  $\mathbf{v}^0$ , premier correcteur dans (v), résout un problème linéaire de nature hyperbolique, à savoir

$$\partial_t \mathbf{v}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{v}^0 + \mathbf{v}^0 \cdot \nabla \mathbf{u}^0 + \nabla p_v^0 = \Delta \mathbf{u}^0, \quad \operatorname{div} \mathbf{v}^0 = 0 \quad \text{dans } Q_{T_0}, \quad (\text{vii})$$

$$\mathbf{v}^0(\cdot, 0) = \mathbf{0} \quad \text{dans } \Omega, \quad (\text{viii})$$

$$\mathbf{v}^0 \cdot \mathbf{n} = 0 \quad \text{sur } \partial\Omega \times (0, T_0). \quad (\text{ix})$$

« Ce problème admet une solution unique  $(\mathbf{v}^0, p_v^0)$ , avec  $\mathbf{v}^0$  dans  $L^\infty(0, T_0; \mathbf{W}_\sigma^{2,2}(\Omega))$  et le contrôle de sa norme en fonction de  $\Omega$ ,  $T^0$  et  $\|\mathbf{u}^0\|_{1,4,2}$  » (voir [2,3]). Les fonctions  $\mathbf{u}^0 + \nu \mathbf{v}^0$  et  $p^0 + \nu p_v^0$  étant maintenant connues dans  $Q_{T_0}$ , on déduit ensuite facilement le problème aux limites non linéaire du type Navier–Stokes avec des conditions aux limites homogènes de Navier qu'il convient de résoudre, pour justifier le second correcteur  $\mathbf{w}^{\nu,\gamma}$  dans (v) et sa pression associée  $p_w^{\nu,\gamma}$ . Alors la même structure vaut pour la pression  $p = p^0 + \nu p_v^0 + \nu p_w^{\nu,\gamma}$ .

L'obtention de bonnes estimations a priori, indépendantes de  $\nu$ , pour  $\mathbf{w}^{\nu,\gamma}$  et  $p_w^{\nu,\gamma}$ , est la partie essentielle de cette Note, d'où les résultats qui seront énoncés dans la partie anglaise, d'une part existence et unicité locales en temps dans l'intervalle  $(0, T_0)$  et pour  $0 \leq \nu \leq \nu^*$  et  $0 \leq \gamma \leq c_1 \nu$ , d'une famille à deux paramètres de solutions fortes pour les problèmes d'Euler ou de Navier–Stokes, d'autre part dépendance continue dans les deux paramètres  $\nu$  et  $\gamma$ .

Le modèle linéaire choisi pour  $\mathbf{v}^0$  est fondamental,  $\Delta \mathbf{u}^0$  ou  $\operatorname{curl}^2 \mathbf{u}^0$  constitue la force externe le « pilotant ». Ainsi défini ce premier correcteur dans (v) assure une subtile transition entre  $\mathbf{u}^0$  la solution du problème d'Euler et les solutions visqueuses du problème  $(\mathbf{P}_{\nu,\gamma})$ , le problème de Navier–Stokes ici considéré avec conditions aux limites de Navier. Nos résultats garantissent, via  $\mathbf{v}^0$  et  $\mathbf{w}^{\nu,\gamma}$ , l'existence de ces solutions sur un intervalle  $(0, T(\nu))$  tel que  $T(\nu) = T_0$ . Ils clarifient le rôle du tenseur de cisaillement sans qu'il soit nécessaire de prendre en compte au voisinage de  $\partial\Omega$  une zone interfaciale artificielle (caractéristique d'une couche limite visqueuse suivant l'idée de Prandtl). Ce rôle est directement lié à la vorticit , nos résultats sont alors plein de sens physique, ils sont finalement en accord avec les remarques classiques de T. Kato [7] (certes dans la situation analysée ici plus simple que celle associée à des conditions de Dirichlet); et ils peuvent aussi impliquer d'intéressantes simulations numériques.

## 1. Formulation of the problem and the main results

There exists an extensive literature on the question whether, in which domains, in which norms and under which conditions solutions of the Navier–Stokes problem tend to solutions of the Euler problem if the viscosity  $\nu$  vanishes. A relatively rich list of references can be found in our papers [3] and [8]. Here we only recall that the convergence of strong solutions of the 3D Navier–Stokes equations to a strong solution of the Euler equations has been so far proven, assuming that the solutions of the Navier–Stokes equations satisfy either 1) the so called generalized impermeability boundary conditions (see [2]), or 2) the so called Navier-type boundary condition (see [3,1,4]), or 3) Navier's boundary condition (see [5]). Note that in [1], the authors work in a 3D cubic domain, considering the homogeneous Navier-type boundary condition on two opposite sides, while they use the conditions of spatial periodicity on the other sides. In [4] and [5], the existence of strong solutions of the Navier–Stokes problem on a time interval independent of  $\nu$  is assumed (i.e. not proven). The authors consider generally inhomogeneous boundary conditions with arbitrary sufficiently smooth right-hand sides. Of many papers, studying related questions in the class of weak solutions, with Navier's boundary condition, we cite e.g. the recent paper [6].

In this Note, we deal with solutions of the Navier–Stokes problem, satisfying inhomogeneous Navier’s boundary condition (with an appropriate “right-hand side”), which is from the physical point of view more realistic than the previously considered generalized impermeability conditions or the Navier-type condition (see [2] or [3]). On the other hand, the study of the Navier–Stokes problem with Navier’s boundary condition brings new difficulties, which did not appear if the other boundary conditions were considered.

Suppose that  $T > 0$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^3$  with the impermeable boundary  $\partial\Omega$  of the class  $C^{3,1}$ . Put  $Q_T := \Omega \times (0, T)$  and  $\Gamma_T := \partial\Omega \times (0, T)$ . We assume that  $\nu \geq 0$  and we deal with the Euler or Navier–Stokes initial-boundary value problem

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_T, \tag{1}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}^* \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_T, \tag{3}$$

supplemented in the case  $\nu > 0$  by the inhomogeneous Navier’s condition

$$[\mathbb{T}_d^v(\mathbf{u}) \cdot \mathbf{n}]_\tau + \gamma \mathbf{u} = \mathbf{a}^{\nu, \gamma} \quad \text{on } \Gamma_T, \tag{4}$$

where function  $\mathbf{a}^{\nu, \gamma}$  is given by (vi) and function  $\mathbf{v}^0$ , appearing in (vi), is a solution of the hyperbolic problem (vii)–(ix). (Here,  $\mathbb{T}_d^v(\mathbf{u})$  is the dynamic stress tensor and the subscript  $\tau$  denotes the tangential component.) The Euler problem (1) <sub>$\nu=0$</sub> –(3) and the problem (vii)–(ix) have unique smooth solutions  $\mathbf{u}^0, \mathbf{v}^0$  on certain time interval  $(0, T_0)$ , see e.g. [9] and [2,3]. In particular,  $\mathbf{v}^0$  satisfies the estimate  $\|\mathbf{v}^0\|_{\infty; 2,2} \leq C$ , where constant  $C$  depends on  $\Omega, T_0$  and  $\|\mathbf{u}^0\|_{1,4,2}$ . (Here,  $\|\cdot\|_{s,m,r}$  denotes the norm in  $L^s(0, T_0; \mathbf{W}^{m,r}(\Omega))$ .)

Using the form (v) of a solution of the Navier–Stokes problem and using also the known information on functions  $\mathbf{u}^0$  and  $\mathbf{v}^0$ , we arrive at the following problem for functions  $\mathbf{w}^{\nu, \gamma}$  and  $p_w^{\nu, \gamma}$ :

$$\partial_t \mathbf{w}^{\nu, \gamma} + \mathbf{A}_1 \mathbf{w}^{\nu, \gamma} + \nu \mathbf{A}_2(\mathbf{w}^{\nu, \gamma}) + \nabla p_w^{\nu, \gamma} = \nu \Delta \mathbf{w}^{\nu, \gamma} + \nu \mathbf{F}^0, \quad \operatorname{div} \mathbf{w}^{\nu, \gamma} = 0 \quad \text{in } Q_{T_0}, \tag{5}$$

$$\mathbf{w}^{\nu, \gamma}(\cdot, 0) = \mathbf{0} \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{w}^{\nu, \gamma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{T_0}, \tag{7}$$

$$[\mathbb{T}_d^v(\mathbf{w}^{\nu, \gamma}) \cdot \mathbf{n}]_\tau + \gamma \mathbf{w}^{\nu, \gamma} = \mathbf{0} \quad \text{on } \Gamma_{T_0}, \tag{8}$$

where the operators  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are defined by the formulas  $\mathbf{A}_1 \mathbf{w}^{\nu, \gamma} := \mathbf{u}^0 \cdot \nabla \mathbf{w}^{\nu, \gamma} + \mathbf{w}^{\nu, \gamma} \cdot \nabla \mathbf{u}^0$ ,  $\mathbf{A}_2(\mathbf{w}^{\nu, \gamma}) := \mathbf{w}^{\nu, \gamma} \cdot \nabla \mathbf{v}^0 + \mathbf{v}^0 \cdot \nabla \mathbf{w}^{\nu, \gamma} + \mathbf{w}^{\nu, \gamma} \cdot \nabla \mathbf{w}^{\nu, \gamma}$  and function  $\mathbf{F}^0$  by  $\mathbf{F}^0 := \Delta \mathbf{v}^0 - \mathbf{v}^0 \cdot \nabla \mathbf{v}^0$ .

The following two theorems represent our main results. Observe that Theorem 1.2 is an immediate consequence of Theorem 1.1.

**Theorem 1.1** (On the nonlinear problem (5)–(8)). *To each  $c_1 > 0$  there exists  $\nu^* > 0$  such that the problem (5)–(8) has a unique solution  $\mathbf{w}^{\nu, \gamma} \in L^\infty(0, T_0; \mathbf{W}^{1,2}(\Omega)) \cap L^2(0, T_0; \mathbf{W}^{2,2}(\Omega))$  for each  $0 < \nu \leq \nu^*$  and  $0 \leq \gamma \leq c_1 \nu$ .*

*Solution  $\mathbf{w}^{\nu, \gamma}$  depends continuously on  $\nu$  and  $\gamma$  in the norm  $\|\cdot\|_{\infty; 1,2} + \|\cdot\|_{2; 2,2}$ .*

*There exist positive constants  $c_2, c_3$  and  $c_4$ , depending on  $\Omega, T_0$  and on the norms  $\|\mathbf{u}^0\|_{1,4,2}$  and  $\|\mathbf{v}^0\|_{\infty; 2,2}$ , however all independent of  $\nu$  and  $\gamma$ , such that*

$$\|\mathbf{w}^{\nu, \gamma}\|_{\infty; 0,2} \leq c_2 \nu, \quad \|\mathbf{w}^{\nu, \gamma}\|_{\infty; 1,2} \leq c_3 \sqrt{\nu}, \quad \|\mathbf{w}^{\nu, \gamma}\|_{2; 2,2} \leq c_4. \tag{9}$$

**Theorem 1.2** (On a family of solutions to the Euler or Navier–Stokes problem). *Suppose that  $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}_\sigma^2(\Omega)$  and  $\mathbf{f} \in L^1(0, T; \mathbf{W}^{4,2}(\Omega))$ . Then there exists  $T_0 \in (0, T]$  such that to each  $c_1 > 0$  there exists  $\nu^* > 0$  and a unique family  $\{\mathbf{u}^{\nu, \gamma}\}$  (for  $0 \leq \nu \leq \nu^*$  and  $0 \leq \gamma \leq c_1 \nu$ ) of solutions of the Euler problem (1) <sub>$\nu=0$</sub> –(3) (if  $\nu = 0$ ) or the Navier–Stokes problem (1)–(4) (if  $0 < \nu \leq \nu^*$  and  $0 \leq \gamma \leq c_1 \nu$ ) in  $L^\infty(0, T_0; \mathbf{W}^{1,2}(\Omega)) \cap L^2(0, T_0, \mathbf{W}^{2,2}(\Omega))$ . (Function  $\mathbf{a}^{\nu, \gamma}$  on the right-hand side of (4) is defined by (vi).)*

*Solution  $\mathbf{u}^{\nu, \gamma}$  has the form (v), where  $\mathbf{v}^0$  is a solution of the linear problem (vii)–(ix) and  $\mathbf{w}^{\nu, \gamma}$  is a solution of the nonlinear problem (5)–(8). The function  $\mathbf{w}^{\nu, \gamma}$  has all the properties, named in Theorem 1.1. Thus, solution  $\mathbf{u}^{\nu, \gamma}$  depends continuously on  $\nu$  and  $\gamma$  in the norm  $\|\cdot\|_{\infty; 1,2} + \|\cdot\|_{2; 2,2}$ .*

## 2. Elements of the proof of Theorem 1.1

In order to simplify the notation, we mostly write only  $\mathbf{w}$  instead of  $\mathbf{w}^{\nu, \gamma}$  in this section. The fundamental part of the proof consists in deriving appropriate a priori estimates. These estimates are basically obtained, testing formally equation (5) by functions  $\mathbf{w}$  and  $-P_\sigma \Delta \mathbf{w}$ . The symbol  $P_\sigma$  denotes the so called Helmholtz projection of  $L^2(\Omega)$  onto the subspace  $\mathbf{L}_\sigma^2(\Omega)$  of divergence-free (in the sense of distributions) functions, having the normal component equal to zero on  $\partial\Omega$  (in the sense of traces). The first step, i.e. the multiplication of (5) by  $\mathbf{w}$  and integration on  $\Omega$ , leads to the estimate

$$\frac{d}{dt} \|\mathbf{w}\|_2^2 + \nu \|\nabla \mathbf{w}\|_2^2 + \gamma \|\mathbf{w}\|_{2;\partial\Omega}^2 \leq (c_5 + c_6\nu) \|\mathbf{w}\|_2^2 + c_7\nu, \tag{10}$$

whose integration yields the energy-type inequality

$$\|\mathbf{w}(\cdot, t)\|_2^2 + \int_0^t [\nu \|\nabla \mathbf{w}\|_2^2 + \gamma \|\mathbf{w}\|_{2;\partial\Omega}^2] \leq \nu c_7 e^{(c_5+c_6\nu)t}. \tag{11}$$

The constants  $c_5, c_6$  and  $c_7$  are independent of  $\nu, \gamma$  and  $T_0$ . The second step, i.e. the multiplication of (5) by  $-P_\sigma \Delta \mathbf{w}$  and integration on  $\Omega$ , provides

$$-\int_\Omega \partial_t \mathbf{w} \cdot P_\sigma \Delta \mathbf{w} + \nu \|P_\sigma \Delta \mathbf{w}\|_2^2 = \int_\Omega \mathbf{A}_1 \mathbf{w} \cdot P_\sigma \Delta \mathbf{w} + \nu \int_\Omega \mathbf{A}_2(\mathbf{w}) \cdot P_\sigma \Delta \mathbf{w} - \nu \int_\Omega \mathbf{F}^0 \cdot P_\sigma \Delta \mathbf{w}. \tag{12}$$

Since  $\partial_t \mathbf{w}(\cdot, t)$  is in  $\mathbf{L}_\sigma^2(\Omega)$ , we can omit  $P_\sigma$  in the integral on the left-hand side. Applying integration by parts and using the boundary conditions (7) and (8), we obtain

$$-\int_\Omega \partial_t \mathbf{w} \cdot P_\sigma \Delta \mathbf{w} = \frac{d}{dt} \|(\nabla \mathbf{w})_s\|_2^2 + \frac{\gamma}{2\nu} \frac{d}{dt} \|\mathbf{w}\|_{2;\partial\Omega}^2, \tag{13}$$

where  $(\nabla \mathbf{w})_s$  is the symmetrized gradient of  $\mathbf{w}$ . The estimate of the first integral on the right-hand side is *one of the crucial points* because we need an estimate with constants independent of  $\nu$  and  $\gamma$ . We use the Helmholtz decomposition  $\Delta \mathbf{w} = P_\sigma \Delta \mathbf{w} + \nabla q$ . Function  $q$  is harmonic and satisfies the Neumann boundary condition  $\partial q / \partial \mathbf{n} = \Delta \mathbf{w} \cdot \mathbf{n}$  on  $\partial\Omega$ . The term  $\Delta \mathbf{w} \cdot \mathbf{n}$  can be successively transformed to the form  $[(\gamma/\nu) \mathbf{curl} \mathbf{w} - 2 \mathbf{curl}(\mathbf{w} \cdot \nabla \mathbf{n})] \cdot \mathbf{n}$  (which contains by one derivative of  $\mathbf{w}$  less than  $\Delta \mathbf{w}$ ), expressing  $\Delta \mathbf{w} \cdot \mathbf{n} = -\mathbf{curl}^2 \mathbf{w} \cdot \mathbf{n} = -\mathbf{curl}[(\mathbf{curl} \mathbf{w})_\tau] = -\mathbf{curl}[\mathbf{n} \times \mathbf{curl} \mathbf{w} \times \mathbf{n}]$ , using the formula  $\nu \mathbf{curl} \mathbf{w} \times \mathbf{n} = [\mathbb{T}_d^v(\mathbf{w}) \cdot \mathbf{n}]_\tau + 2\nu \mathbf{w} \cdot \nabla \mathbf{n}$  (see [1] and [8]) and the boundary condition (8). Applying the known estimates of solutions of the Neumann problem, we derive the inequality  $\|\nabla q\|_2 \leq C(\gamma/\nu) \|\nabla \mathbf{w}\|_2 + C\|\mathbf{w}\|_{1,2}$ . Now, we have

$$\int_\Omega \mathbf{A}_1 \mathbf{w} \cdot P_\sigma \Delta \mathbf{w} = \int_\Omega \mathbf{A}_1 \mathbf{w} \cdot \Delta \mathbf{w} - \int_\Omega \mathbf{A}_1 \mathbf{w} \cdot \nabla q. \tag{14}$$

The second integral on the right-hand side can be estimated, using the mentioned estimate of  $\nabla q$ . In the first integral on the right-hand side, we can integrate by parts. After a lengthy, technical and in some parts also sophisticated procedure (see [8] for details), we arrive at the estimate

$$\int_\Omega \mathbf{A}_1 \mathbf{w} \cdot P_\sigma \Delta \mathbf{w} \equiv \int_\Omega [\mathbf{u}^0 \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}^0] \cdot P_\sigma \Delta \mathbf{w} \leq C \left(1 + \frac{\gamma}{\nu}\right) \|\mathbf{w}\|_{1,2}^2. \tag{15}$$

The right-hand side contains only first order derivatives of function  $\mathbf{w}$ . (Note that the fraction  $\gamma/\nu$  is upper bounded due to the assumption  $\gamma \leq c_1\nu$ .) The estimates of the second and third terms on the right-hand side of (12) are relatively easier, because these terms are multiplied by  $\nu$ . Hence we can apply Schwarz' and Young's inequalities and the arising norms  $\epsilon \|P_\sigma \Delta \mathbf{w}\|_2^2$  (with  $\epsilon$  sufficiently small) can be absorbed by the term  $\nu \|P_\sigma \Delta \mathbf{w}\|_2^2$  on the left-hand side. Thus, we obtain:

$$\begin{aligned} & \frac{d}{dt} \|(\nabla \mathbf{w})_s\|_2^2 + \frac{\gamma}{2\nu} \frac{d}{dt} \|\mathbf{w}\|_{2;\partial\Omega}^2 + \frac{\nu}{2} \|P_\sigma \Delta \mathbf{w}\|_2^2 \\ & \leq C \left(1 + \frac{\gamma}{\nu} + \nu\right) \|\mathbf{w}\|_{1,2}^{*2} + C\nu \|\mathbf{w}\|_{1,2}^{*6} + C\nu \left(1 + \frac{\gamma}{\nu}\right)^{1/2} \|\mathbf{w}\|_{1,2}^{*4} + C\nu, \end{aligned}$$

where  $\|\mathbf{w}\|_{1,2}^* := (\|\mathbf{w}\|_2^2 + \|(\nabla \mathbf{w})_s\|_2^2)^{1/2}$ . Summing this inequality with (10), we get:

$$\begin{aligned} & \frac{d}{dt} \|\mathbf{w}\|_{1,2}^{*2} + \frac{\gamma}{2\nu} \frac{d}{dt} \|\mathbf{w}\|_{2;\partial\Omega}^2 + \nu \|\nabla \mathbf{w}\|_2^2 + \gamma \|\mathbf{w}\|_{2;\partial\Omega}^2 + \frac{\nu}{2} \|P_\sigma \Delta \mathbf{w}\|_2^2 \\ & \leq c_8 \left(1 + \frac{\gamma}{\nu} + \nu\right) \|\mathbf{w}\|_{1,2}^{*2} + c_9\nu \|\mathbf{w}\|_{1,2}^{*6} + c_{10}\nu \left(1 + \frac{\gamma}{\nu}\right)^{1/2} \|\mathbf{w}\|_{1,2}^{*4} + c_{11}\nu, \end{aligned} \tag{16}$$

where the constants  $c_8$ – $c_{11}$  are independent of  $\nu, \gamma$  and  $T_0$ . This inequality can be integrated on the time interval  $(0, T_0)$  for  $0 < \nu \leq \nu^*$ , where  $\nu^*$  is a sufficiently small positive number. In this way, we arrive at the inequalities (9). (We also need to apply Korn's inequality in order to estimate  $\|\nabla \mathbf{w}\|_2$  from above by  $\|(\nabla \mathbf{w})_s\|_2$ .) We profit from the fact that the higher powers of  $\|\mathbf{w}\|_{1,2}^*$  on the right-hand side of (16) are multiplied by  $\nu$ : hence they do not cause the blow up of the resulting inequalities on the fixed time interval  $(0, T_0)$  if  $\nu$  is small enough. Applying e.g. the method of Galerkin approximations and

using a priori estimates (9), we prove the existence of a solution of (5)–(8). The uniqueness of this solution can be proven by contradiction.

Now we return to the notation  $\mathbf{w}^{\nu,\gamma}$  instead of  $\mathbf{w}$ . The proof of continuity of the family  $\{\mathbf{w}^{\nu,\gamma}\}$  in dependence on  $\nu$  and  $\gamma$  in the range  $0 \leq \nu \leq \nu^*$ ,  $0 \leq \gamma \leq c_1 \nu$  is natural: assume that  $\mathbf{w}'$  and  $\mathbf{w}''$  are two solutions, corresponding to  $\nu = \nu'$ ,  $\gamma = \gamma'$  and  $\nu = \nu''$ ,  $\gamma = \gamma''$ . We put  $\mathbf{z} := \mathbf{w}'' - \mathbf{w}'$ . We consider the problem for  $\mathbf{z}$  and estimate the solution (in the norm  $\|\cdot\|_{\infty;1,2} + \|\cdot\|_{2;2,2}$ ) from above by  $C(|\nu'' - \nu'| + |\gamma'' - \gamma'|)$ , where  $C$  is independent of  $\nu$  and  $\gamma$ . The complete procedure is even longer than the derivation of inequalities (9) for function  $\mathbf{w}^{\nu,\gamma}$ , because unlike (8), the boundary condition, satisfied by function  $\mathbf{z}$ , is inhomogeneous. (See [8] for more details.)

### 3. Concluding remarks on the structure of the velocity field

The structure (v) and estimates (9) provide the rate of convergence of solutions  $\mathbf{u}^{\nu,\gamma}$  of the Navier–Stokes problem to solution  $\mathbf{u}^0$  of the Euler problem as  $\nu \rightarrow 0+$ . Due to the assumption  $\gamma \leq c_1 \nu$ , coefficient  $\gamma$  (of friction between the fluid and the wall) also tends to zero when  $\nu$  tends to zero.

From Theorem 1.2 we recall the strategy: starting from the velocity field  $\mathbf{u}^0$  (solution of the Euler problem (1) <sub>$\nu=0$</sub> –(3)) and introducing the first corrector  $\mathbf{v}^0$  as a solution of a linear Euler-type problem (vii)–(ix), we construct a “natural” family  $\{\mathbf{u}^{\nu,\gamma}\}$  of solutions to the incompressible Navier–Stokes equations (1), satisfying the impermeability boundary condition (3) and “appropriately inhomogeneous” Navier’s boundary condition (4). The family is  $\nu$ - and  $\gamma$ -continuous for  $0 \leq \nu \leq \nu^*$  and  $0 \leq \gamma \leq c_1 \nu$  in the topology of strong solutions. Consequently, we answer positively the question of existence of an inviscid limit of solutions  $\mathbf{u}^{\nu,\gamma}$  without any boundary layer in the neighborhood of  $\partial\Omega$ . Slightly different boundary conditions have been considered in some other recent papers: see [8] for discussion and more comments. In this Note, the chosen structure (v) of solutions of the Navier–Stokes problem and the chosen form (vi) of function  $\mathbf{a}^{\nu,\gamma}$  in Navier’s boundary condition (4) (and especially the presence of the corrector  $\mathbf{v}^0$  in (vi) and (4)) enabled us to overcome the difficulties, described in [1] as a challenging problem.

It is remarkable that it is still an open question whether the same approach works, if we impose an appropriate inhomogeneous Dirichlet boundary condition instead of (4) (e.g. the “naive” version  $\mathbf{u}^{\nu,\gamma} = \mathbf{u}^0$  or the more sophisticated variant  $\mathbf{u}^{\nu,\gamma} = \mathbf{u}^0 + \nu \mathbf{v}^0$  on  $\Gamma_{T_0}$ ). It is generally well known that if one considers the Dirichlet boundary condition for solutions of the Navier–Stokes equations then studies of the zero viscosity limit of these solutions represent a difficult problem. The reasons are both mathematically and physically deeper and reflect the fact that Dirichlet’s boundary condition (even “appropriately inhomogeneous”) causes a complicated structure of the velocity field in the neighborhood of  $\partial\Omega$ . Of many papers, dealing with related questions and elucidating the background of this state, we cite T. Kato [7].

It is usually expected that the larger is the viscosity, the smoother are the solutions of the Navier–Stokes equations. However, this expectation has so far not been theoretically confirmed, i.e. the question whether we can arbitrarily increase the upper limit  $\nu^*$  for the viscosity in Theorems 1.1 and 1.2, is open. Similarly, the question whether these theorems hold with the condition  $0 \leq \gamma \leq c_1 \nu^\alpha$  (for some  $\alpha \in (0, 1)$ ) instead of the condition  $0 \leq \gamma \leq c_1 \nu$ , is also open. (The case  $\alpha = 0$  and  $\nu \rightarrow 0$  is closely connected with Dirichlet’s boundary condition for velocity.)

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### References

- [1] H. Beirão da Veiga, F. Crispo, Sharp inviscid limit results under Navier type boundary condition. *An  $L^p$  theory*, J. Math. Fluid Mech. 12 (2010) 397–411.
- [2] H. Bellout, J. Neustupa, A Navier–Stokes approximation of the 3D Euler equation with the zero flux on the boundary, J. Math. Fluid Mech. 10 (2008) 531–553.
- [3] H. Bellout, J. Neustupa, P. Penel, On a  $\nu$ -continuous family of strong solutions to the Euler or Navier–Stokes equations with the Navier-type boundary condition, *Discrete Contin. Dynam. Systems* 27 (4) (2010) 1353–1373.
- [4] G.Q. Chen, D. Osborne, Z. Qian, The Navier–Stokes equations with the kinematic and vorticity boundary conditions on non-flat boundaries, *Acta Math. Sci.* 29 (4) (2009) 919–948.
- [5] G.Q. Chen, Z. Qian, A study of the Navier–Stokes equations with the kinematic and Navier’s boundary conditions, Preprint, December 2008, <http://arxiv.org/pdf/0901.0147>.
- [6] D. Iftimie, G. Planas, Inviscid limit for the Navier–Stokes equations with Navier friction boundary condition, *Nonlinearity* 19 (4) (2006) 899–918.
- [7] T. Kato, Remarks on zero viscosity limit for non-stationary Navier–Stokes flows with boundary, in: S.S. Chern (Ed.), *Seminar on Nonlinear PDE*, in: MSRI, vol. 2, Springer, 1984, pp. 85–98.
- [8] J. Neustupa, P. Penel, On the local in time existence of strong solutions of the Navier–Stokes equations with Navier’s boundary condition and the zero viscosity limit, Preprint USTV-2010.
- [9] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Func. Anal.* 20 (1975) 32–43.