C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1079–1082

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Existence of bound states for the coupled Schrödinger–KdV system with cubic nonlinearity

Existence d'ondes solitaires pour le système couplé de Schrödinger–KdV avec non linearité cubique

João-Paulo Dias [a], Mário Figueira [a], Filipe Oliveira [b]

[a] *CMAF/UL and FCUL, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal*
[b] *Dep. Matemática, FCT/UNL, Monte da Caparica, Portugal*

ARTICLE INFO

Article history:
Received 17 March 2009
Accepted 20 September 2010

Presented by Gilles Lebeau

ABSTRACT

We prove in this Note the existence of an infinite family of smooth positive bound states for the coupled Schrödinger–Korteweg–de Vries system, which decays exponentially at infinity.

RÉSUMÉ

Nous prouvons dans cette Note l'existence d'une famille infinie d'ondes solitaires régulières pour le système couplé de Schrödinger–Korteweg–de Vries, qui décroissent exponentiellement a l'infini.

1. Introduction and main result

Let us consider the coupled Schrödinger–KdV system:

$$\begin{cases} if_t + D^2 f = \beta f g - |f|^2 f, \\ g_t + D^3 g + g D g = \dfrac{\beta}{2} D\big(|f|^2\big), \end{cases} \tag{1}$$

where $f = f(x,t)$ is a complex-valued function, $g = g(x,t)$ is real-valued, $D = \frac{\partial}{\partial x}$ represents the spatial derivative and $\beta < 0$ is a real constant.

The coupled Schrödinger–KdV system appears in the context of interaction phenomena between long waves and short waves such as the resonant interaction between long and short capillary – gravity water waves. The global well-posedness of the Cauchy problem for the I.V.P. associated to (1) was solved recently by A. Corcho and F. Linares [5] in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Here, we are concerned in finding bound-state solutions to (1) of the form

$$\big(f(x,t), g(x,t)\big) = \big(e^{i\omega t} e^{ikx} \phi(x - ct), \psi(x - ct)\big) \tag{2}$$

E-mail addresses: dias@ptmat.fc.ul.pt (J.-P. Dias), figueira@ptmat.fc.ul.pt (M. Figueira), fso@fct.unl.pt (F. Oliveira).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.09.018

where $\phi, \psi \geqslant 0$. By choosing $c = 2k$ and putting $c^* := k^2 + \omega$, we obtain the system

$$\begin{cases} -\phi'' + c^*\phi = \phi^3 - \beta\phi\psi, \\ -\psi'' + c\psi = \dfrac{1}{2}\psi^2 - \dfrac{\beta}{2}\phi^2. \end{cases} \tag{3}$$

In [2], an existence theorem is derived for a general system similar to (3), although the method employed cannot be exploited here due to the absence of a cubic term in the second equation. Also, in [1] and [3], the existence of bound states (and ground states) for the coupled systems

$$\begin{cases} if_t + D^2 f = fg, \\ g_t + \gamma\mathcal{H}Dg = \beta D(|f|^2), \end{cases} \quad \text{and} \quad \begin{cases} i(f_t + c_1 Df) + \delta_1 D^2 f = \alpha fg, \\ g_t + c_2 Dg + D^3 g + \gamma gDg = \beta D(|f|^2) \end{cases} \tag{4}$$

is studied (here, $\mathcal{H}$ denotes de Hilbert transform). Note however that the different approaches used rely heavily on the fact that the nonlinear terms are quadratic and do not seem suitable to handle the term $-|u|^2 u$ in the right-hand side of (1).

In what follows, for $s \in \mathbb{R}$, we denote by $H^s \equiv H^s(\mathbb{R})$ the usual Sobolev space with the norm $\| \, \|_{H^s}$ and the L^p norm will be denoted by $\| \, \|_p$.

We end this introduction by stating our main result:

Theorem 1.1. *For $\beta < -\frac{1}{2}$ there exists a family*

$$(f_n, g_n) = \left(e^{i\omega_n t} e^{ik_n x}\phi_n(x - c_n t), \psi_n(x - c_n t)\right) \tag{5}$$

of non-trivial solutions to (1) *with* $\lim_{n\to\infty} c_n = +\infty$.

Here, ϕ_n, ψ_n are smooth positive functions which decay exponentially at infinity.

2. The minimization problem

For $\mu \geqslant 0$, we set

$$\mathcal{I}(\mu) = \inf\{E(u, v): (u, v) \in X_\mu\}, \tag{6}$$

where $X_\mu = \{(u, v) \in H^1 \times H^1: \|u\|_2^2 + \|v\|_2^2 = \mu\}$ (u, v real-valued) and

$$E(u, v) = \int (Du)^2 + \int (Dv)^2 - \frac{1}{2}\int u^4 - \frac{1}{3}\int v^3 + \beta\int u^2 v. \tag{7}$$

We will apply the concentration–compactness method [6,7] to prove the existence of a minimizer for $\mathcal{I}(\mu)$.

Proposition 2.1. *For all $\mu > 0$, $\mathcal{I}(\mu) > -\infty$.*

Proof. Let $(u, v) \in X_\mu$: $\|u\|_2^2 \leqslant \mu$ and $\|v\|_2^2 \leqslant \mu$. By the Gagliardo–Nirenberg inequalities:

$$\|v\|_3^3 \leqslant C_1\|Dv\|_2^{\frac{1}{2}}\|v\|_2^{\frac{5}{2}} \leqslant C_1\mu^{\frac{5}{4}}\|Dv\|_2^{\frac{1}{2}} \quad \text{and} \quad \|u\|_4^4 \leqslant C_2\|Du\|_2\|u\|_2^3 \leqslant C_2\mu^{\frac{3}{2}}\|Du\|_2,$$

where C_j denote positive constants. Also, $\int |v|u^2 \leqslant \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|u\|_4^4 \leqslant \frac{\mu}{2} + C_2\frac{\mu^{\frac{3}{2}}}{2}\|Du\|_2$. Finally, we obtain

$$\begin{aligned} E(u, v) &\geqslant \|Du\|_2^2 + \|Dv\|_2^2 - \frac{1}{2}\int u^4 - \frac{1}{3}\int |v|^3 - |\beta|\int u^2|v| \\ &\geqslant \|Du\|_2^2 + \|Dv\|_2^2 - C_2(1 + |\beta|)\frac{\mu^{\frac{3}{2}}}{2}\|Du\|_2 - \frac{C_1}{3}\mu^{\frac{5}{4}}\|Dv\|_2^{\frac{1}{2}} - \frac{|\beta|\mu}{2}, \end{aligned} \tag{8}$$

from where we deduce the existence of an inferior bound for $E(u, v)$ depending exclusively on μ. □

Proposition 2.2. *For all $\mu \geqslant 0$, $\mathcal{I}(\mu) \leqslant 0$. Also, there exists $\mu^* > 0$ such that for all $\mu > \mu^*$, $\mathcal{I}(\mu) \leqslant -A\mu^2$, where A is a positive constant independent of μ.*

Proof. Let $\mu \geqslant 0$ and $u \in H^1$ such that $\|u\|_2^2 = \mu$. Then $(u, 0) \in X_\mu$ and $E(u, 0) \leqslant \int (Du)^2$. Noticing that $\inf\{\int (Du)^2: \|u\|_2^2 = \mu\} = 0$, we get $\mathcal{I}_\mu \leqslant 0$.

We now consider $u \in H^1(\mathbb{R})$ such that $\|u\|_2 = 1$. Putting $u_\mu(x) = \mu^{\frac{1}{2}}u(x)$, $(u_\mu, 0) \in X_\mu$. Furthermore,

$$\mathcal{I}(\mu) \leqslant E(u_\mu, 0) = \mu \int (Du)^2 - \frac{1}{2}\mu^2 \int u^4 = \mu\left(\int (Du)^2 - \frac{1}{4}\mu \int u^4\right) - \frac{\mu^2}{4}\int u^4.$$

By choosing $A = \frac{1}{4}\int u^4$ and μ^* such that $\int (Du)^2 - \frac{1}{4}\mu^* \int u^4 \leqslant 0$ we get the result. $\square$

Remark 2.3. It is well known that for $f \in H^1(\mathbb{R})$ real valued, $\|D|f|\|_{L^2} \leqslant \|Df\|_{L^2}$.

For a pair $(u,v) \in X$, $E(|u|,|v|) \leqslant E(u,v)$. Hence, there exists a minimizing sequence (u_j, v_j) for problem (6) with $u_j, v_j \geqslant 0$.

Lemma 2.4. *Let $\mu > \mu^*$. For all $\theta > 1$, $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$.*

Proof. Consider a positive minimizing sequence $(u_j, v_j) \in X_\mu$ for problem (6). We have

$$\begin{aligned} E\big(\sqrt{\theta}u_j, \sqrt{\theta}v_j\big) &= \theta E(u_j, v_j) - \frac{1}{2}\big(\theta^2 - \theta\big)\int u_j^4 + \big(\theta^{\frac{3}{2}} - \theta\big)\left(\beta \int u_j^2 v_j - \frac{1}{3}\int v_j^3\right) \\ &\leqslant \theta E(u_j, v_j) + \max\big\{\theta - \theta^2, \theta - \theta^{\frac{3}{2}}\big\}\left(\frac{1}{2}\int u_j^4 + |\beta|\int u_j^2 v_j + \frac{1}{3}\int v_j^3\right). \end{aligned}$$

Since (u_j, v_j) is a minimizing sequence, $\frac{1}{2}\int u_j^4 + |\beta|\int u_j^2 v_j + \frac{1}{3}\int v_j^3 \geqslant \delta$ for some $\delta > 0$. Otherwise there would exist a subsequence – still denoted (u_j, v_j) – such that $\lim E((u_j, v_j)) \geqslant 0$, which is absurd since $\mathcal{I}(\mu) < 0$. Hence

$$E\big(\sqrt{\theta}u_j, \sqrt{\theta}v_j\big) \leqslant \theta E(u_j, v_j) - \delta\big(\theta^2 - \theta\big) \quad \text{for all } \theta > 1.$$

Since $\|\sqrt{\theta}u_j\|_2^2 + \|\sqrt{\theta}v_j\|_2^2 = \theta\|u_j\|_2^2 + \theta\|v_j\|_2^2 = \theta\mu$, we obtain $\mathcal{I}(\theta\mu) < \theta\mathcal{I}(\mu)$. $\square$

From this lemma it is straightforward to prove that $\mathcal{I}$ is a non-increasing function of μ and therefore there exists $\mu_1 \geqslant 0$ such that $\mathcal{I}(\mu) < 0 \Leftrightarrow \mu > \mu_1$. Arguing as in [8, Lemma 2.3], these facts are sufficient to prove the following key result:

Corollary 2.5 *(Sub-additivity). Let $\mu > \mu_1$ and $0 < \Omega < \mu$. Then $\mathcal{I}(\mu) < \mathcal{I}(\Omega) + \mathcal{I}(\mu - \Omega)$.*

Next, we prove the existence of minimizers:

Proposition 2.6. *Let $\mathcal{M}_\mu = \{(u,v) \in X_\mu\colon \mathcal{I}_\mu = E(u,v)\}$. For $\mu > \mu_1$, $\mathcal{M}_\mu \neq \emptyset$.*

Sketch of the proof. Let us consider a positive minimizing sequence $(u_j, v_j) \in X_\mu$ for problem (6). We will apply the concentration–compactness lemma to the sequence $\rho_j = u_j^2 + v_j^2$. Using the notations in [6], we introduce the concentration function of ρ_j:

$$Q_j(t) = \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} \rho_j, \quad \text{and we set} \quad \Omega = \lim_{t\to\infty} Q(t).$$

We now have three alternatives: vanishing ($\Omega = 0$), dichotomy ($0 < \Omega < \mu$) and compactness ($\Omega = \mu$). The latter implies the relative compactness of the sequence (u_j, v_j) up to translations.

First, we rule out vanishing. Indeed, if $\Omega = 0$, $\lim_{j\to\infty}\sup_{y\in\mathbb{R}}\int_{y-t}^{y+t} u_j^2 = \lim_{j\to\infty}\sup_{y\in\mathbb{R}}\int_{y-t}^{y+t} v_j^2 = 0$.

Since (u_j) and (v_j) are bounded in $H^1(\mathbb{R})$ (as seen in the proof of Proposition 2.1), a classical lemma (see [7, Lemma I.1]) yields $\|u_j\|_p \to 0$ and $\|v_j\|_p \to 0$ for all $p > 2$.

It results that

$$\mathcal{I}(\mu) = \lim_{j\to\infty} E(u_j, v_j) = \lim_{j\to\infty}\int (Du_j)^2 + \int D{v_j}^2 - \frac{1}{2}\int u_j^4 - \frac{1}{3}\int v_j^3 + \beta\int u_j^2 v_j \geqslant 0,$$

which is absurd by Proposition 2.2.

Corollary 2.5 easily rules out dichotomy. Indeed, in the present situation it is standard to construct for all $\epsilon > 0$ two sequences $(u_j^{(i)}, v_j^{(i)})$, $i = 1, 2$, such that

$$\big|\|u_j^{(1)}\|_2^2 + \|v_j^{(1)}\|_2^2 - \Omega\big| < \epsilon, \qquad \big|\|u_j^{(2)}\|_2^2 + \|v_j^{(2)}\|_2^2 - (\mu - \Omega)\big| < \epsilon$$

and

$$E(u_j, v_j) \geqslant E\big(u_j^{(1)}, v_j^{(1)}\big) + E\big(u_j^{(2)}, v_j^{(2)}\big) - C(\epsilon), \qquad \lim_{\epsilon\to 0} C(\epsilon) = 0.$$

This leads to $\mathcal{I}_\mu \geqslant \mathcal{I}_\Omega + \mathcal{I}_{\mu-\Omega}$, which is in contradiction with Corollary 2.5.

Hence, we have compactness: extracting once again a subsequence, there exists $\{y_j\}$ such that

$$\big(\tilde u_j = u_j(.-y_j),\ \tilde v_j = v_j(.-y_j)\big) \to (\phi,\psi) \quad \text{in } L^2(\mathbb{R}).$$

Furthermore, the sequence $(\tilde u_j, \tilde v_j)$ converges to (ϕ,ψ) in $H^1(\mathbb{R})$ weak. Hence, $(\tilde u_j, \tilde v_j) \to (\phi,\psi)$ in L^p for all $p \geqslant 2$: $\|\tilde u_j\|_4 \to \|\phi\|_4$, $\|\tilde v_j\|_3 \to \|\psi\|_3$, $\int \tilde u_j^2 \tilde v_j \to \int \phi^2\psi$ and $\mathcal{I}_\mu \leqslant E(\phi,\psi) \leqslant \underline{\lim}\, E(\tilde u_j, \tilde v_j) = \mathcal{I}_\mu$. Finally, $(\phi,\psi) \in \mathcal{M}_\mu \neq \emptyset$.

Note that we have obtained $\underline{\lim} \int D\tilde u_j^2 + D\tilde v_j^2 = \int D\phi^2 + D\psi^2$, hence the convergence takes place in H^1 strong. Also, it is clear that $\psi \neq 0$. We now show that $\phi \neq 0$: taking ψ such that $(0,\psi) \in X_\mu$, for all $\theta \in [0,1]$, $(\theta^{\frac12}\psi, (1-\theta)^{\frac12}\psi) \in X_\mu$.

A straightforward computation leads to $E(\theta^{\frac12}\psi, (1-\theta)^{\frac12}\psi) \leqslant E(0,\psi) + f_\beta(\theta)\|\psi\|_3^3$, where $f_\beta(\theta) = \frac13(1-(1-\theta)^{\frac32}) + \beta\theta(1-\theta)^{\frac12}$. We get the desired result after observing that $f'_\beta(0) < 0$ for $\beta < -\frac12$, since, for small θ, $E(\theta^{\frac12}\psi, (1-\theta)^{\frac12}\psi) < E(0,\psi)$. $\square$

3. End of the proof of Theorem 1.1

Let $(\phi,\psi) \in \mathcal{M}_\mu$. There exists a Lagrange multiplier $\lambda = \lambda(\mu,\phi,\psi) \in \mathbb{R}$ such that

$$\begin{cases} -\phi'' - \lambda\phi = \phi^3 - \beta\phi\psi, \\ -\psi'' - \lambda\psi = \dfrac12\psi^2 - \dfrac{\beta}{2}\phi^2. \end{cases} \tag{9}$$

Lemma 3.1. *There exists a constant $A > 0$ such that for all $\mu > \mu^*$, $\lambda(\mu,\phi,\psi) \leqslant -A\mu$.*

Proof. Multiplying Eqs. (9) by ϕ and ψ respectively and integrating leads to

$$\int D\phi^2 + \int D\psi^2 - \lambda\mu = \int \phi^4 - \frac{3\beta}{2}\int \phi^2\psi + \frac12\int \psi^3.$$

Since $\mathcal{I}_\mu = E(\phi,\psi) = \int \phi'^2 + \int \psi'^2 - \frac12\int\phi^4 - \frac13\int\psi^3 + \beta\int\phi^2\psi$, we get

$$\lambda\mu = \mathcal{I}_\mu - \frac12\int\phi^4 + \frac{\beta}{2}\int\phi^2\psi - \frac16\int\psi^3 \leqslant \mathcal{I}_\mu \leqslant -A\mu^2, \quad \text{by Proposition 2.2.}$$

The proof of the main theorem is complete by choosing a sequence $\mu_n \to \infty$ and setting $c_n = -\lambda_n$, $k_n = -\frac12\lambda_n$ and $\omega_n = -\lambda_n - k_n^2$. Note that the functions ϕ, ψ are positive since they are the limit in H^1 of a positive minimizing sequence. Also, a classical bootstrap argument proves the regularity of ϕ and ψ. Finally, since $c_n^* = k_n^2 + \omega_n = -\lambda_n > 0$, the argument used in the proof of Theorem 8.1.1 in [4] (see also [1, Theorem 2.1]) easily proves the existence of $\epsilon_1, \epsilon_2 > 0$ such that $e^{\epsilon_1|x|}\phi(x), e^{\epsilon_2|x|}\psi(x) \in L^\infty$, which results in the exponential decreasing of ϕ and ψ at infinity. $\square$

Remark 3.2. In the particular case where $0 > \beta > -\frac16$ and $c = 4c^* - \frac{1}{12}\beta(1+6\beta)$, it is possible to exhibit explicit solutions: noticing that $D^2\operatorname{sech} = \operatorname{sech} - 2\operatorname{sech}^3$ and $D^2\operatorname{sech}^2 = 4\operatorname{sech}^2 - 6\operatorname{sech}^4$, it is easy to verify that the system (3) possesses the exact solution

$$\phi(x) = \frac{\sqrt{2c^*(1+6\beta)}}{\cosh(\sqrt{c^*}x)}, \qquad \psi(x) = \frac{12c^*}{\cosh^2(\sqrt{c^*}x)}.$$

Acknowledgements

The research of J.P. Dias and M. Figueira was partially supported by FCT, Financiamento Base 2008-ISFL-1-209 and the research of F. Oliveira was partially supported by Financiamento Base 2008-ISFL-1-297.

References

[1] J. Albert, J. Angulo Pava, Existence and stability of ground-state solutions of a Schrödinger–KdV system, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 987–1029.
[2] A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 453–458.
[3] J. Angulo Pava, J.F. Montenegro, Existence and evenness of solitary-wave solutions for an equation of short and long dispersive waves, Nonlinearity 13 (2000) 1595–1611.
[4] T. Cazenave, An Introduction to Nonlinear Schrödinger Equations, Textos de Métodos Matemáticos, vol. 22, Instituto de Matemática, UFRJ, Rio de Janeiro, 1989.
[5] A.J. Corcho, F. Linares, Well-posedness for the Schrödinger–Korteweg–de Vries system, Trans. Amer. Math. Soc. 359 (2007) 4089–4106.
[6] P.L. Lions, The concentration-compactness principle in the calculus of variations, Part 1, Ann. Inst. H. Poincaré 1 (1984) 109–145.
[7] P.L. Lions, The concentration-compactness principle in the calculus of variations, Part 2, Ann. Inst. H. Poincaré 1 (1984) 223–283.
[8] M. Ohta, Stability of stationary states for the coupled Klein–Gordon–Schrödinger equations, Nonlinear Anal. TMA 27 (1996) 455–461.