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Expansion of the Green's function for divergence form operators

Expansion de la fonction de Green pour les opérateurs de type divergence

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ABSTRACT

We consider the fundamental solution G_a of the operator $-\Delta_a = -\frac{1}{a(x)}\operatorname{div}(a(x)\nabla\cdot)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$, associated to the Dirichlet boundary condition, where a is a positive smooth function on $\overline{\Omega}$. In this short Note, we give a precise description of the function $G_a(x,y)$. In particular, we define in a unique way its continuous part $H_a(x,y)$ and we prove that the corresponding Robin's function $R_a(x) = H_a(x,x)$ belongs to $C^\infty(\Omega)$, although $H_a \notin C^1(\Omega \times \Omega)$ in general.

RÉSUMÉ

On considère la solution fondamentale G_a de l'opérateur $-\Delta_a = -\frac{1}{a(x)}\operatorname{div}(a(x)\nabla\cdot)$, sur un domaine borné régulier $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ avec les conditions de Dirichlet au bord, ici a est une fonction régulière et strictement positive sur $\overline{\Omega}$. Dans cette Note, on donne une description précise de la fonction $G_a(x,y)$. On définit notamment $H_a(x,y)$, la partie continue de G_a et on montre que la fonction de Robin correspondante $R_a(x) = H_a(x,x)$ est dans $C^\infty(\Omega)$, sachant que $H_a \notin C^1(\Omega \times \Omega)$ en général.

Version française abrégée

Soient Ω un domaine borné régulier de $\mathbb{R}^n$ avec $n \geqslant 2$ et a une fonction strictement positive et C^∞ sur $\overline{\Omega}$. On considère l'opérateur elliptique suivant :

$$\Delta_a u = \frac{1}{a(x)}\operatorname{div}\big(a(x)\nabla u\big) = \Delta u + \nabla \log a \cdot \nabla u.$$

On note par $G_a(x,y)$ la fonction de Green de l'opérateur $-\Delta_a$ avec les conditions de Dirichlet au bord, i.e. $\forall y \in \Omega$,

$$\begin{cases} -\Delta_a G_a(x,y) = \delta_y & \text{sur } \Omega, \\ G_a(x,y) = 0 & \text{sur } \partial\Omega. \end{cases}$$

Dans la cas isotrope, i.e. quand a est une constante, l'opérateur Δ_a est le laplacien usuel. La fonction de Green correspondante G_Δ s'écrit $G_\Delta(x,y) = \Phi_0(x-y) + H_\Delta(x,y)$, où

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$$\Phi_0(x) = -\frac{1}{2\pi}\log|x|, \quad \text{si } n = 2; \qquad \Phi_0(x) = \frac{1}{n(n-2)\omega_n}|x|^{2-n}, \quad \text{si } n \geqslant 3; \tag{1}$$

avec ω_n le volume de la boule unité de $\mathbb{R}^n$. On sait que H_Δ est appelée la partie régulière de G_Δ, car $H_\Delta \in C^\infty(\Omega\times\Omega)$. On en déduit que la fonction de Robin $R_\Delta(x) = H_\Delta(x,x) \in C^\infty(\Omega)$. De plus, $H_\Delta(x,y) = H_\Delta(y,x)$ pour tout $x, y \in \Omega$.

La fonction de Green joue un rôle essentiel dans beaucoup de domaines en mathématiques, en particulier pour les phénomènes de concentration des équations aux dérivées partielles elliptiques (voir [2,1,14]). En fait, dans les phénomènes d'explosion, les singularités sont souvent localisées par des fonctionnelles associées à la fonction de Green (voir par exemple [15,12,14,8,13]). D'autre part, quand on veut montrer l'existence de ces solutions à limite singulière, il est fondamental de bien comprendre le comportement de la fonction de Green, puisque les solutions approchées sont souvent construites en utilisant les perturbations de la fonction de Green. D'ailleurs, la régularité C^1 de la fonction de Robin associée est souvent nécessaire pour appliquer les arguments de perturbation et les techniques variationnelles (voir [5,16]).

L'objectif de cette note est d'explorer le cas anisotrope i.e. quand a est non constante. Soit $H(x,y) = G_a(x,y) - \Phi_0(x-y)$,

$$-\Delta_a H(x,y) = \nabla\log a(x)\cdot\nabla\Phi_0(x-y) = -\frac{1}{n\omega_n}\nabla\log a(x)\cdot\frac{x-y}{|x-y|^n} \quad \text{dans } \Omega.$$

Si $\nabla a(y) \neq 0$, on voit que $-\Delta_a H \notin L^2(\Omega)$ pour tout $n \geqslant 2$, donc l'application $x \mapsto H(x,y)$ n'est pas dans $H^2(\Omega)$. D'ailleurs, la fonction H n'est plus symétrique car $a(y)G_a(x,y) = a(x)G_a(y,x)$ pour $x \neq y \in \Omega$. C'est bien connu que G_a et G_Δ sont comparables (voir [10,7] pour $n \geqslant 3$ et [9,4] pour $n = 2$), mais il n'existe pas d'expansion précise pour G_a. Toutefois, on signale qu'en dimension 2, quelques résultats ont été établis (voir [3,16]), comme le Lemme 2.1 dans [16] qui montre que $x \mapsto H(x,x)$ est dans $C^1(\Omega)$ quand $n = 2$. On cite aussi [11] pour les résultats récents sur les fonctions de Green des opérateurs qui ne sont pas sous forme de divergence.

Dans ce travail, on donne une expansion détaillée de G_a (voir Théorèmes 2.6 et 3.3). Ces formules nous permettent de définir d'une façon unique H_a, la partie régulière de G_a. De plus, on montre que la fonction de Robin $x \mapsto R_a(x) = H_a(x,x) \in C^\infty(\Omega)$, bien que $H_a \notin C^1(\Omega\times\Omega)$ en général. L'expansion de G_a dépend de la parité de la dimension n, l'idée principale est de décomposer la fonction G_a dans des sous espaces de fonction bien précis. Notre résultat permet aussi de bien comprendre les dérivées de G_a à n'importe quel ordre.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n \geqslant 2$ and a be a positive function in $C^\infty(\overline{\Omega})$. Consider the following elliptic operator:

$$\Delta_a u = \frac{1}{a(x)}\operatorname{div}\big(a(x)\nabla u\big) = \Delta u + \nabla\log a\cdot\nabla u.$$

Let $G_a(x,y)$ be the Green's function of $-\Delta_a$ associated to the Dirichlet boundary condition, i.e. $\forall y \in \Omega$,

$$\begin{cases} -\Delta_a G_a(x,y) = \delta_y & \text{in } \Omega,\\ G_a(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

For the isotropic case $a \equiv$ *positive constant*, the operator Δ_a is just the usual Laplacian. The corresponding Green's function G_Δ is written as $G_\Delta(x,y) = \Phi_0(x-y) + H_\Delta(x,y)$ where

$$\Phi_0(x) = -\frac{1}{2\pi}\log|x|, \quad \text{if } n = 2; \qquad \Phi_0(x) = \frac{1}{n(n-2)\omega_n}|x|^{2-n}, \quad \text{if } n \geqslant 3; \tag{2}$$

with ω_n the Lebesgue volume of the unit ball in $\mathbb{R}^n$. It is well known (see [6]) that H_Δ, called the regular part of G_Δ, is smooth in $\Omega\times\Omega$ and symmetric (i.e. $H_\Delta(x,y) = H_\Delta(y,x)$ for any $x, y \in \Omega$). Hence $R_\Delta(x) = H_\Delta(x,x)$, the so called Robin's function is smooth in Ω.

The understanding of the Green's function for elliptic operators plays an essential role in many fields of mathematics (see [2] and references therein), especially in the study of concentration phenomena for semilinear or quasilinear partial differential equations (see [1,14] and sequel works). More precisely, for many blow-up phenomena, the location of possible singularities is frequently determined by functionals associated to the corresponding Green's function and its regular part (see for example [15,12,14,8,13]). Conversely, when we try to build such singular limiting solutions, it is necessary to understand well the behavior of Green's function, since the approximate solutions are often constructed by perturbations of the Green's function. Moreover, the C^1 regularity of the corresponding Robin's function is also necessary for handling perturbation arguments and variational techniques (see for example [5,16]).

If R_Δ is well known, the situation turns out to be less clear for the anisotropic case, i.e. when a is not constant. Let $H(x,y) = G_a(x,y) - \Phi_0(x-y)$,

$$-\Delta_a H(x,y) = \nabla\log a(x)\cdot\nabla\Phi_0(x-y) = -\frac{1}{n\omega_n}\nabla\log a(x)\cdot\frac{x-y}{|x-y|^n} \quad \text{in } \Omega.$$

If $\nabla a(y) \neq 0$, we see that $-\Delta_a H \notin L^2(\Omega)$ for any $n \geqslant 2$, so $x \mapsto H(x, y)$ does not belong to $H^2(\Omega)$. Furthermore, we have no longer the symmetry of H, since $a(y)G_a(x, y) = a(x)G_a(y, x)$ for $x \neq y \in \Omega$. It is well known that for divergence form operators, G_a is comparable with G_Δ (see [10,7] for $n \geqslant 3$ and [9,4] for $n = 2$). But as far as we are aware, there does not exist in the literature any precise expansion of G_a with anisotropic a. Some special results exist in dimension two (see [3,16]). For example, it was proved in Lemma 2.1 of [16] that $x \mapsto H(x, x)$ belongs to $C^1(\Omega)$ when $n = 2$. It is also worth mentioning that the divergence form operator cases are usually considered easier than operators in nondivergence form, where many efforts have been done (see the recent work [11] and references therein).

In this work, we give a general expansion of G_a (see Theorems 2.6 and 3.3 below), which permits us to define in a unique way the regular part H_a. Furthermore, we prove that the Robin's function $x \mapsto R_a(x) = H_a(x, x) \in C^\infty(\Omega)$, although H_a does not belong to $C^1(\Omega \times \Omega)$ in general. Our expansion of G_a depends on the parity of the dimension n. In the sequel, we first focus on the odd dimensional case, then we can state more easily the even dimensional situation, where the decomposition of G_a requires some additional logarithmic terms.

2. The case of odd dimension

We introduce the following notations. For any $\alpha = (\alpha_i) \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, let $|\alpha| = \sum_{1 \leqslant i \leqslant n} \alpha_i$ and $x^\alpha = \Pi_{i=1}^n x_i^{\alpha_i}$. The symbol γ denotes always an arbitrary constant in $(0, 1)$. For any subsets A_j in a vector space V, we set $A_0 + A_1 + \cdots + A_p = \{\sum_{0 \leqslant j \leqslant p} v_j,\ v_j \in A_j\}$. We write $A_0 \oplus A_1 \oplus \cdots \oplus A_p$ when the sum is direct.

Definition 2.1. Given $k \in \mathbb{N}^*$, let $E_k = \operatorname{span}\{x^\alpha r^{-n}, |\alpha| = k, \alpha \in \mathbb{N}^n\}$ where $r = \|x\|$ is the Euclidean norm. Let $E_{k,m} = \operatorname{span}\{x^\alpha r^{2m-n}, |\alpha| = k - 2m\}$ for any $m \in \mathbb{N}$, $2m \leqslant k$.

Lemma 2.2. *Let n be odd, then for any $k \in \mathbb{N}^*$, $\Delta : E_{k+2} \to E_k$ is bijective.*

Proof. Obviously, $E_k \subset L^1_{loc}$ for $k \geqslant 1$. By direct calculations, it is easy to see that the operator Δ maps E_{k+2} into E_k. We first prove the surjectivity.

Clearly, we have $E_k = \sum_{0 \leqslant 2m \leqslant k} E_{k,m}$. Given $f(x) = x^\alpha r^{2m-n} \in E_{k,m}$, let $g(x) = r^2 f \in E_{k+2,m+1}$, we get (using $x \cdot \nabla(x^\beta) = |\beta| x^\beta$, $\forall \beta \in \mathbb{N}^n$)

$$\Delta g = \Delta\big(x^\alpha r^{2m+2-n}\big) = r^{2m+2-n}\Delta\big(x^\alpha\big) + 2(2m + 2 - n)\big(|\alpha| + m\big) f.$$

As n is odd and $|\alpha| + m \neq 0$ (since $|\alpha| + 2m = k > 0$), $\lambda = 2(2m + 2 - n)(|\alpha| + m) \neq 0$, we obtain

$$\lambda^{-1}\Delta g - f = r^{2m+2-n}\Delta\big(x^\alpha\big) \in E_{k,m+1}.$$

When $|\alpha| \leqslant 1$, we already have $\Delta(\lambda^{-1} g) = f$, i.e. $f \in \Delta(E_{k+2})$. When $|\alpha| \geqslant 1$, we conclude by decreasing induction on $|\alpha|$ as $\Delta(x^\alpha) \in \operatorname{span}\{x^\beta, |\beta| = |\alpha| - 2\}$. On the other hand, as the only homogeneous harmonic functions are homogeneous harmonic polynomials, they cannot be in $E_\ell \setminus \{0\}$ for any $\ell \in \mathbb{N}^*$ since n is odd, and the map is injective. □

Lemma 2.3. *Let n be odd, Ω_0 be a bounded smooth domain in $\mathbb{R}^n$ containing the origin 0 and c be a smooth function defined on $\overline{\Omega}_0$. For any $f \in E_k$, $k \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, there exists a unique*

$$g \in E_{k+2} \oplus E_{k+3} \oplus \cdots \oplus E_{n+\ell} \oplus C^{\ell,\gamma}(\overline{\Omega}_0)$$

such that $-\Delta g = c(x) f$ in Ω_0 and $g = 0$ on $\partial\Omega_0$.

Remark 2.4. If $k \geqslant n + \ell - 1$, by $E_{k+2} \oplus E_{k+3} \oplus \cdots \oplus E_{n+\ell}$, we mean just $\{0\}$.

Proof. We set $x^{(j)} = (x, x, \ldots, x) \in \Omega_0^j$. By Taylor's formula,

$$c(x) = \sum_{j=0}^{n+\ell-k} \frac{1}{j!}\nabla^j c(0) \cdot x^{(j)} + \frac{1}{(n + \ell - k)!}\int_0^1 \big[(1 - t)^{n+\ell-k}\nabla^{n+\ell-k+1} c(tx)\big]\,dt \cdot x^{(n+\ell-k+1)}.$$

Therefore

$$c(x) f \in E_k \oplus E_{k+1} \oplus \cdots \oplus E_{n+\ell} \oplus C^{\ell,\gamma}(\overline{\Omega}_0).$$

As the trace of any function in E_j on $\partial\Omega_0$ is smooth, using Lemma 2.2, we get a unique solution

$$g \in E_{k+2} \oplus E_{k+3} \oplus \cdots \oplus E_{n+\ell+2} \oplus C^{\ell+2,\gamma}(\overline{\Omega}_0).$$

Since $E_{n+\ell+1} \oplus E_{n+\ell+2} \subset C^{\ell,\gamma}(\overline{\Omega}_0)$, we are done. □

Now we generalize the above result for anisotropic operators:

Proposition 2.5. *Let n, Ω_0 and c be as in Lemma 2.3. Assume that b is a smooth positive function on $\overline{\Omega}_0$. Then, for any $f \in E_k$, $k \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, there exists a unique*

$$\xi \in E_{k+2} \oplus E_{k+3} \oplus \cdots \oplus E_{n+\ell} \oplus C^{\ell,\gamma}(\overline{\Omega}_0)$$

such that $-\Delta_b \xi = c(x) f$ in Ω_0 and $\xi = 0$ on $\partial\Omega_0$.

Proof. Consider the equation $-\Delta_b \xi = f$. There exists a sequence of functions ξ_j such that $-\Delta\xi_0 = f$, $-\Delta\xi_{j+1} = \nabla \log b \cdot \nabla\xi_j$, and $\xi_j = 0$ on $\partial\Omega_0$ for $0 \leqslant j \leqslant n + \ell - k$. By induction, using Lemma 2.3 with $(\ell + 1)$, we can claim that

$$\xi_j \in E_{k+j+2} \oplus E_{k+j+3} \oplus \cdots \oplus E_{n+\ell+1} \oplus C^{\ell+1,\gamma}(\overline{\Omega}_0), \quad \forall 0 \leqslant j \leqslant n + \ell - k.$$

In particular, $\xi_{n+\ell-k} \in C^{\ell+1,\gamma}(\overline{\Omega}_0)$. Finally, we solve $-\mathrm{div}(b\nabla h) = \nabla b \cdot \nabla\xi_{n+\ell-k}$ in Ω_0 with Dirichlet boundary condition. The unique solution h belongs to $C^{\ell+2,\gamma}(\overline{\Omega}_0)$ by standard elliptic theory. Thus

$$\xi = h + \sum_{j=0}^{n+\ell-k} \xi_j \in E_{k+2} \oplus E_{k+3} \oplus \cdots \oplus E_{n+\ell} \oplus C^{\ell,\gamma}(\overline{\Omega}_0)$$

satisfies $-\Delta_b \xi = f$ in Ω_0 and $\xi = 0$ on $\partial\Omega_0$. The proof of the general case $-\Delta_b \xi = cf$ uses just the idea of Taylor expansion as in Lemma 2.3. We leave the details to the reader. □

We are now able to state the main result:

Theorem 2.6. *Let n be odd. For any $\ell \in \mathbb{N}$, there exist unique functions $\Phi_k \in E_{k+2}$ depending on $y \in \Omega$ for $0 \leqslant k \leqslant n + \ell - 2$, and $H^\ell \in C^{\ell,\gamma}(\overline{\Omega} \times \Omega)$ such that*

$$G_a(x, y) = \sum_{0 \leqslant k \leqslant n+\ell-2} \Phi_k(x - y) + H^\ell(x, y) \quad \text{in } \overline{\Omega} \times \Omega. \tag{3}$$

Proof. Fix $M > 0$ large enough such that $\overline{\Omega}_y \subset B_M$ for any $y \in \Omega$, where $\Omega_y = \Omega(\cdot - y)$ and B_M is the ball centered at 0 with radius M. We also extend a to a positive smooth function in $\mathbb{R}^n$ (this is possible since Ω is smooth). Let $y \in \Omega$ and Φ_0 be as in (2), so $\Phi_0 \in E_2$ (recall that $n \geqslant 3$). For any $\ell \in \mathbb{N}$, we use Proposition 2.5 with $\Omega_0 = B_M$ to get a solution ξ_y such that

$$\xi_y \in E_3 \oplus E_4 \oplus \cdots \oplus E_{n+\ell} \oplus C^{\ell,\gamma}(\overline{B}_M),$$

and $-\Delta_{a_y} \xi_y = \nabla \log(a_y) \cdot \nabla\Phi_0$ in B_M, $\xi_y = 0$ on ∂B_M. Here $a_y(x) = a(x + y)$.

Then $R_y(x) = G_a(x, y) - \Phi_0(x - y) - \xi_y(x - y)$ is smooth on $\overline{\Omega}$, since it verifies $-\Delta_a R_y = 0$ in Ω with smooth boundary value on $\partial\Omega$. This yields the expansion

$$G_a(x, y) = \sum_{0 \leqslant k \leqslant n+\ell-2} \Phi_k(x - y) + H^\ell(x, y) \quad \text{in } \overline{\Omega} \times \Omega,$$

with $\Phi_k \in E_{k+2}$ and $x \mapsto H^\ell(x, y) \in C^{\ell,\gamma}(\overline{\Omega})$. Moreover, checking carefully the dependence on y in each step of the construction, we can conclude that $y \mapsto H^\ell(\cdot, y) \in C^\infty(\Omega, C^{\ell,\gamma}(\overline{\Omega}))$, which implies $H^\ell \in C^{\ell,\gamma}(\overline{\Omega} \times \Omega)$. □

Therefore, we define the regular part of G_a and the corresponding Robin's function R_a as

$$H_a(x, y) = H^0(x, y) \quad \text{and} \quad R_a(x) = H_a(x, x).$$

By definition, H_a is a continuous function over $\overline{\Omega} \times \Omega$. Furthermore, for any $\ell \in \mathbb{N}^*$,

$$H_a(x, y) = \sum_{n-1 \leqslant k \leqslant n+\ell-2} \Phi_k(x - y) + H^\ell(x, y) \quad \forall (x, y) \in \overline{\Omega} \times \Omega. \tag{4}$$

An interesting consequence is the following:

Proposition 2.7. *The Robin's function R_a is smooth in Ω, as a and Ω are smooth.*

Proof. For any $\ell \in \mathbb{N}$, using expansion (4), we see that $R_a(x) = H^\ell(x, x)$ since $\Phi_k(0) = 0$ for any $k \geqslant n - 1$. Hence for any $\ell \in \mathbb{N}$, $R_a \in C^\ell(\Omega)$ by the regularity of H^ℓ. □

3. The case of even dimension

To handle the even dimensional case, we introduce some further notations since logarithmic terms appear in the expansion:

Lemma 3.1. *Given $k \in \mathbb{N}$, let $L_k = \operatorname{span}\{x^\alpha \log r, |\alpha| = k, \alpha \in \mathbb{N}^n\}$. Then for any $k \in \mathbb{N}$, $L_k \subset \Delta(L_{k+2} \oplus \mathbb{R}[x])$ where $\mathbb{R}[x]$ is the set of real polynomials with variables x_i.*

Proof. Let $m \in \mathbb{N}$, $2m \leqslant k$, $L_{k,m} = \operatorname{span}\{x^\alpha r^{2m} \log r, |\alpha| = k - 2m\}$, so $L_k = \sum_{0 \leqslant 2m \leqslant k} L_{k,m}$. For $f = x^\alpha r^{2m} \log r \in L_{k,m}$, consider $g = r^2 f \in L_{k+1,m+1}$, then we have

$$\Delta g = r^{2m+2} \log r \Delta\big(x^\alpha\big) + \big(2|\alpha| + n + 4m + 2\big) x^\alpha r^{2m} + (2m+2)\big(2m + 2|\alpha| + n\big) x^\alpha r^{2m} \log r.$$

Since $(2m+2)(2m+2|\alpha|+n) > 0$ and $x^\alpha r^{2m} \in \mathbb{R}[x]$, we obtain $\widetilde{g} \in L_{k+2} \oplus \mathbb{R}[x]$ such that $f - \Delta\widetilde{g} = r^{2m+2} \log r \Delta(x^\alpha) \in L_{k,m+1}$. As above, the proof is completed by decreasing induction on $|\alpha|$. $\square$

Let $n \in 2\mathbb{N}^*$, $k \in \mathbb{N}^*$ and E_k be as in Definition 2.1. When $k \geqslant n$, we denote by E_k^s the *singular subset* of E_k, i.e. $E_k^s = E_k \setminus \mathbb{R}[x]$. Therefore $E_k \subset E_k^s \oplus \mathbb{R}[x]$ for any $k \geqslant n$.

Lemma 3.2. *Let $n \in 2\mathbb{N}^*$, $k \in \mathbb{N}^*$,*

$$F_k = \begin{cases} E_k & \text{if } k < n, \\ E_k^s \oplus L_{k-n} & \text{if } k \geqslant n \end{cases} \quad \text{and} \quad \widetilde{E}_k = \begin{cases} F_k & \text{if } k < n, \\ F_k \oplus \mathbb{R}[x] & \text{if } k \geqslant n. \end{cases}$$

Then $\widetilde{E}_k \subset \Delta(\widetilde{E}_{k+2})$ for any $k \in \mathbb{N}^*$.

Proof. If $k < n-2$, let $f = x^\alpha r^{2m-n} \in E_{k,m}$, since $2m \leqslant k < n-2$, we can just repeat the proof of Lemma 2.2 with decreasing induction on $|\alpha|$, so $E_k \subset \Delta(E_{k+2}) \subset \Delta(\widetilde{E}_{k+2})$.

When $k = n-2$ or $n-1$, we need to handle the case $2m = n-2$, i.e. for $f = x^\alpha r^{-2}$ with $|\alpha| = (k - n + 2) \leqslant 1$. Let $g = x^\alpha \log r \in L_{k-n+2} \subset \widetilde{E}_{k+2}$, then $\Delta g = (2|\alpha| + n - 2) x^\alpha r^{-2}$ (when $n = 2$, only the case $|\alpha| = 1$ occurs as $k \geqslant 1$), so we get the claim.

For $k \geqslant n$, the case $f \in L_{k-n} \oplus \mathbb{R}[x]$ is treated by Lemma 3.1. Let $f \in E_k^s$ be as in the previous case, when we apply decreasing induction on $|\alpha|$, we need to pay attention for the case $2m = n-2$. For $f_0 = x^\alpha r^{2m} r^{-n} = x^\alpha r^{-2} \in E_{k-n+2}$, using again $g = x^\alpha \log r$,

$$\Delta g = \big(2|\alpha| + n - 2\big) x^\alpha r^{-2} + \log r \times \Delta\big(x^\alpha\big),$$

hence there exists $g_0 \in L_{k-n+2}$ such that $(\Delta g_0 - f_0) \in L_{k-n}$. We conclude by Lemma 3.1. $\square$

Replacing E_k by F_k, we can proceed in the same way as for Theorem 2.6 to get the expansion for the even dimensional situation.

Theorem 3.3. *Let $n \in 2\mathbb{N}^*$. For any $\ell \in \mathbb{N}$, there exist unique functions $\Phi_k \in F_{k+2}$ depending on $y \in \Omega$ for $0 \leqslant k \leqslant n + \ell - 2$ and $H^\ell \in C^{\ell,\gamma}(\overline{\Omega} \times \Omega)$ such that*

$$G_a(x, y) = \sum_{0 \leqslant k \leqslant n+\ell-2} \Phi_k(x - y) + H^\ell(x, y) \quad \text{in } \overline{\Omega} \times \Omega. \tag{5}$$

As for the odd dimensional case, we define the regular part of G_a by H^0 and conclude that the Robin's function $x \mapsto R_a(x) = H_a(x, x)$ is smooth in Ω, this explains why we need not precise that n is odd for Proposition 2.7.

Of course, it is now easy to expand the Green's function $G_{\mathcal{L}}$ for more general operators as $\mathcal{L}_{a,b} u = -\operatorname{div}(a(x)\nabla u)/b(x)$, for which $G_{\mathcal{L}_{a,b}}(x, y) = a(y) G_a(x, y)/b(y)$. We can also deduce the expansion of any derivative of the function G_a or $G_{\mathcal{L}}$. Lastly, we note that the smoothness of a and Ω is used to obtain a smooth Robin's function R_a, but if we just need to get $R_a \in C^1(\Omega)$, the assumptions on a and Ω can be greatly weakened.

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