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The Camassa–Holm equation on the half-line with linearizable boundary condition

L'équation de Camassa–Holm sur la demi-droite avec condition aux limites linéarisable

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A R T I C L E I N F O

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A B S T R A C T

We present a Riemann–Hilbert problem formalism for the initial boundary value problem for the Camassa–Holm equation on the half-line $x > 0$ with homogeneous Dirichlet boundary condition at $x = 0$. We show that, similarly to the problem on the whole line, the solution of this problem can be obtained in parametric form via the solution of a Riemann–Hilbert problem determined only by the initial data via associated spectral functions. This allows us to apply the nonlinear steepest descent method and to describe the large-time asymptotics of the solution. There are three sectors of the quarter plane $x > 0$, $t > 0$ where the asymptotic behavior is qualitatively different.

R É S U M É

Nous considérons un problème aux limites pour l'équation de Camassa–Holm sur la demi-droite $x > 0$ avec condition de Dirichlet homogène au bord $x = 0$. Nous montrons que, comme dans le cas du problème de Cauchy sur la droite, la solution $u(x,t)$ s'exprime, sous forme paramétrique, en termes de la solution d'un problème de Riemann–Hilbert auxiliaire, entièrement déterminé par des fonctions spectrales associées aux seules données initiales. Cela permet d'appliquer la méthode de plus grande descente non linéaire et d'obtenir ainsi le comportement asymptotique de la solution pour les grandes valeurs du temps. Cette analyse met en évidence trois secteurs du quadrant $x > 0$, $t > 0$ où la solution a des comportements asymptotiques de types différents.

Version française abrégée

L'objet de cette Note est de déterminer le comportement asymptotique, lorsque $t \to +\infty$, de la solution $u(x,t)$ du problème aux limites (1) pour l'équation de Camassa–Holm sur la demi-droite. Nous procédons par scattering inverse, en reformulant le problème en termes d'un problème de Riemann–Hilbert matriciel. Nous supposons que la donnée initiale $u_0(x)$ est une fonction suffisamment lisse, qui tend assez rapidement vers 0 lorsque $|x| \to +\infty$ et qui vérifie $u_0(x) - u_{0xx}(x) + 1 > 0$ pour tout $x > 0$. Sous ces hypothèses nous obtenons une représentation de la solution $u(x,t)$ du problème aux limites en termes de la solution d'un problème de Riemann–Hilbert associé. Les données d'un tel problème sont définies

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dans le plan complexe du « paramètre spectral » k. Elles consistent en une condition de saut définie par une « matrice de saut », le long d'un contour de ce plan, ainsi qu'en des conditions de résidu en un nombre fini de points. Les données du problème de Riemann–Hilbert que nous construisons sont exprimées en termes de fonctions spectrales associées à la donnée initiale u_0.

En général, l'application à un problème aux limites de ce type de méthode, i.e., la construction d'un problème de Riemann–Hilbert associé, requiert la connaissance de données au bord excédentaires. Autrement dit, elle oblige à considérer un problème aux limites surdéterminé [3,7].

Il existe néanmoins des données au bord particulières, dites « linéarisables » [15], pour lesquelles la reformulation en termes d'un problème de Riemann–Hilbert associé n'utilise pas de données au bord excédentaires, ce qui permet de rester dans le cadre d'un problème aux limites bien posé. La construction que nous proposons ici montre que, pour l'équation de Camassa–Holm sur la demi-droite, la condition (1c) est « linéarisable ».

La représentation de la solution du problème aux limites (1) que nous obtenons ainsi nous permet d'appliquer la méthode du « col non linéaire » [12] pour étudier son comportement asymptotique pour de grandes valeurs du temps. Cette étude révèle qu'il y a trois domaines du quart de plan $x > 0$, $t > 0$ dans lesquels la solution a un comportement asymptotique de types différents.

1. Introduction

We consider the initial–boundary value problem for the Camassa–Holm (CH) equation [8] on the half-line with homogeneous boundary condition:

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x > 0,\ t > 0, \tag{1a}$$

$$u(x,0) = u_0(x), \quad x > 0, \tag{1b}$$

$$u(0,t) = 0, \quad t > 0, \tag{1c}$$

where $\omega > 0$ is a parameter and $u_0(x)$ is assumed to be sufficiently smooth, fast decaying as $x \to +\infty$ and satisfies $u_0(0) = 0$.

The CH equation is a model equation describing the shallow-water approximation in inviscid hydrodynamics, where the constant ω is related to the critical shallow water wave speed.

Our approach is based on the adaptation of the inverse scattering method developed in [5] for the initial value problem for the CH equation. A new method for analysing initial–boundary value (IBV) problems based on ideas of the inverse scattering transform was introduced in 1997 by Fokas [14] (see also [15]) and was further developed by several authors (see for example [2,3,17,18]). This method expresses the solution in terms of the solution of a matrix RH problem for which it is possible to obtain useful asymptotic information applying the nonlinear steepest descent method [12] (for the CH equation on the whole line, see [4,6]).

This RH problem involves some unknown spectral functions which can be in principle characterised by the so-called global relation. A breakthrough for analysing the global relation was discovered by the authors and Fokas [1,16]. For the CH equation, the compatibility of the boundary values $u(0,t)$, $u_x(0,t)$, and $u_{xx}(0,t)$ (together with $u(x,0)$), which are required for the construction of a full set of spectral functions, is analysed, in spectral terms, in [7].

It was shown in [15] that for a particular class of boundary conditions, called linearizable, it is possible to bypass the global relation and to solve the relevant initial–boundary value problem with the same level of efficiency as the Cauchy problem. Linearizable IBV problems are discussed in [2,20,19].

In this paper we show that the IBV problem (1) is of this type: its solution can be expressed in terms of the solution of a RH problem, whose jump matrix is determined in terms of spectral functions associated with the initial data $u_0(x)$ only, and, in turn, the asymptotics of $u(x,t)$ as $t \to +\infty$ can be given in terms of these spectral functions.

Notice that for $\omega \neq 0$, one cannot convert the IBV problem (1) into the Cauchy problem on the whole line, by setting $u(x,t) = -u(-x,t)$ for $x < 0$ as in the case $\omega = 0$, cf. [13].

2. The Riemann–Hilbert problem

In what follows we assume, without loss of generality, that $\omega = 1$. We also assume that $u_0(x)$ satisfies a positivity condition: for all $x > 0$, $m_0(x) + 1 > 0$, where $m_0(x) := u_0(x) - u_{0xx}(x)$. Then, as in the case of the Cauchy problem on the whole line [10,11], one can show that the IBV problem (1) has a solution $u(x,t)$ for all $t > 0$. Moreover, this solution is unique [9] and satisfies $m(x,t) + 1 > 0$, where $m := u - u_{xx}$.

Given $m_0(x)$, we define the spectral functions $a(k)$ and $b(k)$, $\operatorname{Im} k \geqslant 0$, by $\binom{b(k)}{a(k)} = \Psi(0,k)$, where $\Psi(x,k)$ is the solution of the differential equation

$$\Psi_x(x,k) = \left(-\mathrm{i}k\sqrt{m_0(x)+1}\,\sigma_3 + \frac{1}{4}\frac{m_{0x}(x)}{m_0(x)+1}\sigma_1 + \frac{1}{8\mathrm{i}k}\frac{m_0(x)}{\sqrt{m_0(x)+1}}(\mathrm{i}\sigma_2 + \sigma_3)\right)\Psi(x,k) \tag{2}$$

satisfying the condition $\Psi(x,k) = \left(\begin{smallmatrix} 0 \\ e^{ikc} \end{smallmatrix}\right)(1+o(1))$ as $x \to +\infty$, where $c = \int_0^\infty (\sqrt{m_0(\xi)+1} - 1)\,d\xi$. Here σ_j, $j = 1,2,3$ are the Pauli matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Theorem 2.1. *The solution $u(x,t)$ of the IBV problem* (1) *can be expressed, in parametric form, in terms of the solution $\mu(y,t,k) = (\mu_1\ \mu_2)(y,t,k)$ of the vector-valued Riemann–Hilbert problem*

$$\begin{aligned} &\mu_-(y,t,k) = \mu_+(y,t,k)J(y,t,k), \quad k \in \Sigma, \\ &\mu(y,t,k) \to (1\ 1), \quad k \to \infty, \\ &\mu_1(y,t,-k) = \mu_2(y,t,k), \\ &\operatorname{Res}_{k=i\nu_j} \mu_1(y,t,k) = i\gamma_j e^{-2\nu_j(y-\frac{2t}{1-4\nu_j^2})} \mu_2(y,t,i\nu_j). \end{aligned} \tag{3}$$

The input data for this problem – the contour Σ, the jump matrix J and the residue parameters ν_j, γ_j – are defined in terms of spectral functions uniquely determined by the spectral functions $a(k)$ and $b(k)$ associated with the initial data $u_0(x)$. More precisely:

(i) *The contour is $\Sigma = \mathbb{R} \cup \{|k| = \frac{1}{2}\} \cup \{|k - \frac{i}{2}| = \varepsilon,\ |k| > \frac{1}{2}\} \cup \{|k + \frac{i}{2}| = \varepsilon,\ |k| > \frac{1}{2}\}$ with a small positive ε.*

(ii) *The jump matrix is*

$$J(y,t,k) = e^{-ik(y-\frac{2t}{1+4k^2})\sigma_3} J_0(k) e^{ik(y-\frac{2t}{1+4k^2})\sigma_3}, \quad k \in \Sigma, \tag{4}$$

where

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & \bar{r}(k) \\ -r(k) & 1-|r(k)|^2 \end{pmatrix}, & \operatorname{Im} k = 0,\ |k| > \frac{1}{2}, \\ \begin{pmatrix} 1 & 0 \\ X(k) & 1 \end{pmatrix}, & |k| = \frac{1}{2},\ |k - \frac{i}{2}| < \varepsilon, \\ \begin{pmatrix} 1 & 0 \\ -H(k) & 1 \end{pmatrix}, & |k| > \frac{1}{2},\ |k - \frac{i}{2}| = \varepsilon, \\ \begin{pmatrix} 1 & 0 \\ X(k)-H(k) & 1 \end{pmatrix}, & |k| = \frac{1}{2},\ |k - \frac{i}{2}| > \varepsilon,\ \operatorname{Im} k > 0, \\ \begin{pmatrix} 1 & 0 \\ -r(k)+X(k)-H(k) & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{r}(k)-\bar{X}(k)+\bar{H}(k) \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} k = 0,\ |k| < \frac{1}{2}, \end{cases}$$

and $J_0(k) = \sigma_1 \overline{J_0(\bar{k})}^{-1} \sigma_1$ for $\operatorname{Im} k < 0$. Here $r(k)$, $X(k)$, and $H(k)$ are expressed in terms of $a(k)$ and $b(k)$ as follows:

$$r(k) = -\frac{\bar{b}(k)}{a(k)}, \qquad H(k) = \frac{\kappa_0 - \kappa_0^{-1}}{2a(k)\tilde{a}(k)} e^{-ikc}, \qquad X(k) = -\frac{\bar{\tilde{b}}(1/4\bar{k}) e^{-2ikc}}{\tilde{a}(k)\Delta(k)}, \tag{5}$$

where $\kappa_0 = (m_0(0)+1)^{1/4}$, $\Delta(k) = \tilde{a}(k)\bar{\tilde{a}}(\frac{1}{4k}) - \tilde{b}(k)\bar{\tilde{b}}(\frac{1}{4k})$, and

$$\begin{aligned} \tilde{a}(k) &= \frac{1}{2}\left\{\left(\kappa_0 + \frac{1}{\kappa_0}\right)a(k) - \left(\kappa_0 - \frac{1}{\kappa_0}\right)b(k)\right\} e^{-ikc}, \\ \tilde{b}(k) &= \frac{1}{2}\left\{\left(\kappa_0 + \frac{1}{\kappa_0}\right)b(k) - \left(\kappa_0 - \frac{1}{\kappa_0}\right)a(k)\right\} e^{-ikc}. \end{aligned} \tag{6}$$

(iii) *The residue parameters $\{\nu_j, \gamma_j\}_{j=1}^N$ are as follows: the $i\nu_j$'s, $0 < \nu_j < \frac{1}{2}$ are the zeros of $\Delta(k)$ (generically, they are not zeros of $\tilde{a}$), and*

$$\gamma_j = -i\frac{\bar{\tilde{b}}(i/4\nu_j)}{\tilde{a}(i\nu_j)\dot{\Delta}(i\nu_j)} > 0. \tag{7}$$

(iv) *$u(x,t)$ is expressed in terms of the solution $\mu(y,t,k)$ of the RH problem* (3) *evaluated as $k \to \frac{i}{2}$ by:*

$$x(y,t) = y + \ln\frac{\mu_1(y,t,\frac{i}{2})}{\mu_2(y,t,\frac{i}{2})}, \qquad u(y,t) = \frac{1}{2i}\lim_{k\to\frac{i}{2}}\left(\frac{\mu_1(y,t,k)\mu_2(y,t,k)}{\mu_1(y,t,\frac{i}{2})\mu_2(y,t,\frac{i}{2})} - 1\right)\frac{1}{k-\frac{i}{2}}. \tag{8}$$

The construction of the RH problem (3) follows from

(a) the relations, in the complex plane of the spectral parameter, amongst the dedicated solutions (eigenfunctions) of the Lax pair equations associated with the CH equation,

(b) a special symmetry of the "t-equation" of the Lax pair, provided by the boundary condition (1c).

• Regarding (a), it is useful to re-express the original Lax pair [8]

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m+1)\psi, \qquad \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi \tag{9}$$

as vector-valued, first order equations [5,7]

$$\Psi_x(x,t,k) = U(x,t,k)\Psi(x,t,k), \tag{10a}$$

$$\Psi_t(x,t,k) = V(x,t,k)\Psi(x,t,k) \tag{10b}$$

involving a new spectral parameter k defined by $\lambda = -k^2 - \frac{1}{4}$. Namely, with $m = m(x,t)$,

$$U = -\mathrm{i}k\sqrt{m+1}\,\sigma_3 + \frac{1}{4}\frac{m_x}{m+1}\sigma_1 + \frac{1}{8\mathrm{i}k}\frac{m}{\sqrt{m+1}}(\mathrm{i}\sigma_2 + \sigma_3),$$

$$V = \mathrm{i}ku\sqrt{m+1}\sigma_3 + \left[\frac{m_t}{4(m+1)} + \frac{u_x}{2}\right]\sigma_1 - \frac{1}{8\mathrm{i}k}\frac{u(m+2)}{\sqrt{m+1}}(\mathrm{i}\sigma_2 + \sigma_3)$$
$$- \frac{\mathrm{i}k}{1+4k^2}\left\{\sqrt{m+1}(\mathrm{i}\sigma_2 - \sigma_3) - \frac{1}{\sqrt{m+1}}(\mathrm{i}\sigma_2 + \sigma_3)\right\}.$$

Eqs. (10) are convenient to control the large-k behavior of its solutions whereas in order to control their behavior as $k \to \pm\frac{\mathrm{i}}{2}$ (which corresponds to $\lambda \to 0$) it is convenient to use a gauge equivalent system

$$\Psi_{0x}(x,t,k) = U_0(x,t,k)\Psi_0(x,t,k), \tag{11a}$$

$$\Psi_{0t}(x,t,k) = V_0(x,t,k)\Psi_0(x,t,k), \tag{11b}$$

where

$$U_0 = -\mathrm{i}k\sigma_3 + \frac{1+4k^2}{8\mathrm{i}k}m(\mathrm{i}\sigma_2 + \sigma_3),$$

$$V_0 = \frac{2\mathrm{i}k}{1+4k^2}\sigma_3 + \frac{u_x}{2}\sigma_1 - \frac{u}{4k}\sigma_2 + \frac{1+4k^2}{8\mathrm{i}k}u\{2\sigma_3 + m(\mathrm{i}\sigma_2 + \sigma_3)\}.$$

The spectral functions $\tilde{a}(k)$ and $\tilde{b}(k)$, $\operatorname{Im} k \geqslant 0$, are defined, similarly to $a(k)$ and $b(k)$, in terms of the Jost solution of Eq. (11a) for $t = 0$ whereas the pairs of spectral functions $\tilde{A}(k)$ and $\tilde{B}(k)$, and $A(k)$ and $B(k)$, respectively are defined, for $k \in \{\operatorname{Im} k \geqslant 0,\ |k| \geqslant \frac{1}{2}\} \cup \{\operatorname{Im} k \leqslant 0,\ |k| \leqslant \frac{1}{2}\}$, in terms of the Jost solutions of (10b) and (11b) for $x = 0$, respectively. In particular, $A(k;T)$, $B(k;T)$, $\tilde{A}(k;T)$, and $\tilde{B}(k;T)$ are determined, in general, by the three functions involved in (10b) and (11b): $u(0,t)$, $u_x(0,t)$, and $u_{xx}(0,t)$ for $t \in [0,T]$. Then, the scattering relationships amongst the Jost-type solutions of (10) and (11) can be written [7] in the form of a RH problem whose data are given in terms of the above spectral functions. For instance, for $|k| = \frac{1}{2}$ the jump matrix is as in (4) with

$$X = -\frac{\overline{B(\bar{k})}/\overline{A(\bar{k})}}{\tilde{a}(k)(\tilde{a}(k) - \tilde{b}(k)\overline{B(\bar{k})}/\overline{A(\bar{k})})}\mathrm{e}^{-2\mathrm{i}kc}.$$

On the other hand, the consistency of the boundary values and the initial values $u(x,0)$ can be expressed, in spectral terms, in the form of the global relation:

$$\tilde{a}(k)B(k;T) - A(k;T)\tilde{b}(k) = \mathrm{O}\left(\left(k - \frac{\mathrm{i}}{2}\right)\mathrm{e}^{-\frac{4\mathrm{i}k}{1+4k^2}T}\right) \quad \text{if} \quad \operatorname{Im} k > 0 \text{ and } |k| > \frac{1}{2}. \tag{12}$$

• Regarding (b) we notice that if $u = 0$, then (11b) (determining $A(k)$ and $B(k)$) reduces to the Zakharov–Shabat system with potential $u_x/2$:

$$\Psi_{0t} + \mathrm{i}\tilde{k}\sigma_3\Psi_0 = \frac{u_x}{2}\sigma_1\Psi_0 \quad \text{with } \tilde{k} = -\left(2k + \frac{1}{2k}\right)^{-1}. \tag{13}$$

Hence $A(k)$ and $B(k)$ satisfy the symmetry condition

$$A(1/4k) = A(k), \qquad B(1/4k) = B(k). \tag{14}$$

Then (12) and (14) suggest to define $R(k)$ in terms of the spectral functions $\tilde{a}$ and $\tilde{b}$:

$$R(k) := \begin{cases} \frac{\tilde{b}(k)}{\tilde{a}(k)}, & \operatorname{Im} k \geqslant 0,\ |k| \geqslant \frac{1}{2}, \\ \frac{\tilde{b}(1/4k)}{\tilde{a}(1/4k)}, & \operatorname{Im} k \leqslant 0,\ |k| \leqslant \frac{1}{2}, \end{cases} \tag{15}$$

and, consequently, to define the jump matrix on the whole contour as well as the residue conditions in terms of the initial data only, substituting $B(k)/A(k)$ by $R(k)$; this leads to the RH problem (3). Then one can show that $u(x,t)$ determined by (8) satisfies: (i) the CH equation; (ii) the initial condition (1b); and (iii) the boundary condition (1c) for $t \in [0,T]$ for all $T > 0$.

3. Long-time asymptotics

The nonlinear steepest descent method for studying the long-time behavior of solutions of oscillating RH problems (involving the time as a large parameter) [12] makes crucial use of the associated "signature table", i.e., the partition of the k-plane into domains with respect to the sign of $\operatorname{Im}\theta$, where θ enters the exponentials in the jump matrix J in the form $\mathrm{e}^{\pm \mathrm{i}t\theta}$. For the CH equation we have: $J = \mathrm{e}^{-\mathrm{i}t\theta\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}t\theta\sigma_3}$, where

$$\theta = \theta(\zeta, k) = \zeta k - \frac{2k}{1+4k^2} \quad \text{with } \zeta = \frac{y}{t}. \tag{16}$$

The implementation of the method involves a series of contour deformations as well as jump matrix transformations aimed at reducing the original RH problem to one with decaying, as $t \to +\infty$, jump matrix (to the identity matrix); then, tracing back the transformations one is able to obtain the long-time asymptotics of the solution of the original problem. With this view, it is important that the jump matrices on the parts of the contour outside the real axis, see (4), have a trigonal structure so that the entries involving the exponentials $\mathrm{e}^{\pm \mathrm{i}t\theta}$ decay to 0. This fact allows us to basically follow the long-time analysis for the initial value problem for the CH equation, see [6,4], which leads to the following

Theorem 3.1. *The quarter plane $x > 0, t > 0$ can be divided into domains where the long-time behavior of the solution $u(x,t)$ of the IBV problem* (1) *has qualitatively different character. Namely,*

(i) *For $x/t > 2 + \varepsilon$ with any $\varepsilon > 0$, the solitons dominate the asymptotics: let $v_j = \frac{2}{1-4\nu_j^2}$, then*

$$u(x,t) = \sum_{j=1}^{N} F_j\big(Y_j(X)\big)|_{X = x - v_j t - x_{0j}} + \mathrm{o}(1),$$

where

$$F_j(Y) = \frac{16\nu_j^2}{1-4\nu_j^2} \frac{1}{1+4\nu_j^2 + (1-4\nu_j^2)\cosh(2\nu_j Y)}, \qquad X_j(Y) = Y + \log \frac{1-2\nu_j + (1+2\nu_j)\mathrm{e}^{-2\nu_j Y}}{1+2\nu_j + (1-2\nu_j)\mathrm{e}^{-2\nu_j Y}},$$

$$x_{0j} = \frac{1}{2\nu_j} \log \frac{\gamma_j}{2\nu_j} + \frac{1}{\nu_j} \sum_{l=j+1}^{N} \log \frac{\nu_l - \nu_j}{\nu_l + \nu_j} + \log \frac{1+2\nu_j}{1-2\nu_j} + 2 \sum_{l=j+1}^{N} \log \frac{1+2\nu_l}{1-2\nu_l} - c,$$

$0 < \nu_1 < \cdots < \nu_N < \frac{1}{2}$ *are such that $\Delta(\mathrm{i}\nu_j) = 0$, see* (6), *and $\{\gamma_j\}_{j=1}^{N}$ are given in* (7).

(ii) *For $\varepsilon < x/t < 2 - \varepsilon$, the asymptotics is described by decaying modulated oscillations*

$$u(x,t) = \frac{c_1}{\sqrt{t}} \sin(c_2 t + c_3 \log t + c_4)\big(1 + \mathrm{o}(1)\big),$$

where c_j, $j = 1, \dots, 4$ are functions of x/t, cf. [6,4].

(iii) *For $|\frac{x}{t} - 2| t^{2/3} < C$,*

$$u(x,t) = -\left(\frac{4}{3t}\right)^{2/3} \big(v^2(s) - v_s(s)\big)\big(1 + \mathrm{o}(1)\big) \quad \textit{with } s = 6^{-1/3}\left(\frac{x}{t} - 2\right) t^{2/3},$$

where $v(s)$ is the solution of the Painlevé II equation $v_{ss}(s) = s v(s) + 2v^3(s)$ fixed by the asymptotics $v(s) \sim \frac{\tilde{b}(0)}{2\sqrt{\pi}\tilde{a}(0)} s^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3}s^{3/2}}$ as $s \to +\infty$.

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