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Méthodes de Galerkine centrées aux mailles

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ABSTRACT

In this Note we propose a new approach to obtain and analyze lowest order methods for diffusive problems yielding at the same time convergence rates and convergence to minimal regularity solutions. The approach merges ideas from Finite Volume and discontinuous Galerkin methods.

RÉSUMÉ

Dans cette Note, on propose une nouvelle approche pour obtenir et analyser des méthodes d'ordre bas pour les problèmes diffusifs. L'analyse permet d'estimer à la fois le taux de convergence et de prouver la convergence vers des solutions à régularité minimale. Cette approche combine les avancées récentes dans les méthodes de Volumes Finis et de Galerkine discontinues.

Version française abrégée

Dans cette Note on propose une nouvelle approche pour obtenir et analyser des méthodes d'ordre bas pour les problème diffusifs. Les idées sont présentées pour le problème modèle

$$-\nabla\cdot(\nu\nabla u)=f \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{1}$$

où Ω denote un ouvert polygonal borné, $f\in L^2(\Omega)$ et ν est un tenseur uniformement elliptique et constant par morceaux sur une partition P_Ω de Ω. La méthode est composée de plusieurs étapes : (i) pour toute face du maillage, on définit un interpolateur de trace consistant pour des fonctions suffisamment régulières ; (ii) à partir des valeurs de trace, on reconstruit un gradient par maille qui est aussi consistant ; (iii) le gradient permet définir un sous-espace de l'espace des fonctions affines par morceaux, qui est ensuite utilisé dans une formulation inspirée des méthodes de Galerkine discontinues (dG). Pour des solutions suffisamment régulières, la forme bilinéaire discrète s'étend à l'espace conténant la solution, ce qui permet de bénéficier des techniques d'analyse par estimation d'erreur propres aux méthodes dG. D'autre part, les techniques d'analyse par compacité typiques des Volumes Finis (cf., e.g., Eymard, Gallouët et Herbin [5]) s'appliquent aussi en vertu des résultats récemment obtenus pour les espaces polynomiaux par morceaux dans [4]. Par simplicité, on utilisera dans la présentation l'interpolateur barycentrique proposé dans [5]. Cet interpolateur, simple à mettre en œuvre, ne tient toutefois pas compte des hétérogénéités du tenseur de diffusion. Parmi les autres choix possible pour mieux gérer les opérateurs hétérogène, on signale l'interpolateur en L introduit par Aavatsmark et al. [1] et récemment analysé dans [2]. A titre d'exemple, la méthode sera appliquée aussi à un problème de diffusion–advection-réaction.

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1. Discretization

Letting $a(u,v) \stackrel{\text{def}}{=} \int_\Omega \nu \nabla u \cdot \nabla v$, the weak formulation of problem (1) reads

$$\text{Find} \quad u \in H_0^1(\Omega) \quad \text{s.t.} \quad a(u,v) = \int_\Omega f v \quad \text{for all } v \in H_0^1(\Omega). \tag{2}$$

Let $\mathcal{H} \subset \mathbb{R}_+ \setminus \{0\}$ be a countable set having 0 as an accumulation point and let $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ define an admissible polyhedral mesh family in the sense of [4] parametrized by the meshsize h and compatible with P_Ω. Let $h \in \mathcal{H}$ be given. For each $T \in \mathcal{T}_h$, let $x_T \in T \setminus \partial T$ be such that T is star-shaped with respect to x_T (the *cell center*) and, for all faces $F \in \mathcal{F}_h$, set $\hat{x}_F \stackrel{\text{def}}{=} \int_F x/|F|$. The set of faces of an element $T \in \mathcal{T}_h$ will be denoted by $\mathcal{F}_h^T$, the set of boundary faces by $\mathcal{F}_h^b$. For each internal face we select an arbitrary but fixed direction for the normal $\mathrm{n}_F \equiv \mathrm{n}_{T_1,F}$ and denote by T_1 the element out of which n_F points. The diameters of the generic element $T \in \mathcal{T}_h$ and face $F \in \mathcal{F}_h$ are denoted by h_T and h_F respectively.

Let $\mathbb{V}^{\mathcal{T}_h} \stackrel{\text{def}}{=} \mathbb{R}^{\mathrm{card}(\mathcal{T}_h)}$, $\mathbb{V}^{\mathcal{F}_h} \stackrel{\text{def}}{=} \mathbb{R}^{\mathrm{card}(\mathcal{F}_h)}$. We call a *trace interpolator* $\mathbf{I}$ a linear bounded operator from $\mathbb{V}^{\mathcal{T}_h}$ to $\mathbb{X} \subset \mathbb{V}^{\mathcal{F}_h}$ such that, for all $\mathbb{V}^{\mathcal{T}_h} \ni \mathbf{V} = \{v_T\}_{T\in\mathcal{T}_h}$, $\mathbb{X} \ni \mathbf{I}(\mathbf{V}) = \{I_F(\mathbf{V})\}_{F\in\mathcal{F}_h}$ is such that $I_F(\mathbf{V}) = 0$ if $F \in \mathcal{F}_h^b$. To ease the presentation, we shall henceforth refer to the barycentric interpolator of [5] defined hereafter. Let, for each $F \in \mathcal{F}_h^i$, $\mathcal{I}_h^F \subset \mathcal{T}_h$ be such that $\mathrm{card}(\mathcal{I}_h^F) = d$ and denote by $\{\alpha_T^F\}_{T\in\mathcal{I}_h^F}$ the family of reals such that $\hat{x}_F = \sum_{T\in\mathcal{I}_h^F} \alpha_T^F x_T$. The barycentric interpolator is defined as follows:

$$\forall \mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}, \quad I_F(\mathbf{V}) = \begin{cases} \sum_{T\in\mathcal{I}_h^F} \alpha_T^F v_T & \text{if } F \in \mathcal{F}_h^i, \\ 0 & \text{if } F \in \mathcal{F}_h^b. \end{cases} \tag{3}$$

To ensure the consistency of the method, the following property must be fulfilled by the trace interpolator for any function w in a suitable space V:

$$\forall T \in \mathcal{T}_h,\ \forall F \in \mathcal{F}_h^T, \quad \left| w(\hat{x}_F) - I_F\left(\Pi_{\mathbb{V}_h^{\mathcal{T}}}(w)\right)\right| \leqslant C_w h_F^2, \tag{4}$$

where $\Pi_{\mathbb{V}^{\mathcal{T}_h}} : V \to \mathbb{V}^{\mathcal{T}_h}$ is the operator that maps every element $w \in V$ onto $\Pi_{\mathbb{V}^{\mathcal{T}_h}}(w) \stackrel{\text{def}}{=} \{w(x_T)\}_{T\in\mathcal{T}_h}$. For the barycentric interpolator (3) we can take

$$V \stackrel{\text{def}}{=} \left\{ w \in C_0^2(\overline{\Omega});\ \nu \nabla w \in H(\mathrm{div};\Omega) \right\}.$$

The global regularity assumptions in V are only needed to prove optimal convergence rates and strongly depend on the choice of the interpolator. When ν is heterogeneous, the L-interpolator introduced in [1] and analyzed in [2] may be preferable, as only local regularity inside each element of P_Ω is required. Convergence to minimal regularity solutions that are barely in $H_0^1(\Omega)$ is also obtained, but no estimate of the convergence rate is available in this case.

The trace interpolator is the key ingredient to construct a consistent gradient approximation. For all $\mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}$, we let (cf. [5])

$$\mathbf{G}_T(\mathbf{V}) \stackrel{\text{def}}{=} \frac{1}{|T|} \sum_{F\in\mathcal{F}_h^T} |F| \left(I_F(\mathbf{V}) - v_T \right) \mathrm{n}_{T,F}.$$

For $k \geqslant 0$, let $P_h^k \stackrel{\text{def}}{=} \{p_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ p_{h|T} \in \mathbb{P}_d^k(T)\}$. Consider the following approximation space:

$$V_h \stackrel{\text{def}}{=} \left\{ v_h \in P_h^1;\ \exists \mathbf{V} \in \mathbb{V}^{\mathcal{T}_h},\ \forall T \in \mathcal{T}_h,\ v_h(x_T) = v_T \text{ and } \nabla v_{h|T} = \mathbf{G}_T(\mathbf{V}) \right\}.$$

For every $v_h \in V_h$, there holds $v_{h|T}(x) = v_T + \mathbf{G}_T(\mathbf{V}) \cdot (x - x_T)$ for all $T \in \mathcal{T}_h$ and $\mathbf{V}$ the element of $\mathbb{V}^{\mathcal{T}_h}$ associated to v_h. The space V_h is thus a subspace of P_h^1 with piecewise affine test functions whose value at x_T is prescribed and whose gradient depends on the values at neighbouring cell centers, on the mesh and, possibly, on the problem data via the interpolator $\mathbf{I}$. We define the following norm on the augmented space $V(h) \stackrel{\text{def}}{=} V + V_h$: For all $v \in V(h)$,

$$\|v\|_h^2 \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{[L^2(T)]^d}^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \left\| [\![v]\!] \right\|_{L^2(F)}^2,$$

where, for all φ such that a (possibly two-valued) trace is available on F,

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \begin{cases} \varphi_{|T_1} - \varphi_{|T_2} & \text{for all } F \in \mathcal{F}_h^{T_1} \cap \mathcal{F}_h^{T_2}, \\ \varphi_{|T} & \text{for all } F \in \mathcal{F}_h^T \cap \mathcal{F}_h^b. \end{cases}$$

The Poincaré inequality proved in [4] for the piecewise broken affine space ensures that $\|\cdot\|_{\mathrm{dG}}$ is a norm.

Lemma 1 *(Approximation estimate). For all $w \in V$ with elementwise bounded gradient, there exists C_w independent of the meshsize h such that*

$$\inf_{w_h \in V_h} \|w - w_h\|_h \leqslant C_w h. \tag{5}$$

Proof. Use (4) and proceed as in [5, Theorem 4.2] to prove that

$$\sum_{T \in \mathcal{T}_h} \|\nabla w_h - \nabla w\|^2_{[L^2(T)]^d} \leqslant C_w h^2.$$

For the jump term, observe that, since w is continuous across mesh interfaces and it vanishes on $\partial\Omega$, $[\![w_h]\!] = [\![w_h - w]\!]$ for all $x \in F$, $F \in \mathcal{F}_h$. Using Agmon's trace inequality together with Taylor expansion yields the desired result. □

Consider the following bilinear form inspired from the SIPG method of Arnold: For all $(w, v_h) \in V(h) \times V_h$

$$a_h(w, v_h) \stackrel{\text{def}}{=} \int_\Omega \nu\nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F \big[\{\nu\nabla_h w\}\cdot \mathrm{n}_F [\![v_h]\!] + [\![w]\!]\{\nu\nabla_h v_h\}\cdot \mathrm{n}_F\big] + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F \mathrm{n}_F \nu\cdot \mathrm{n}_F [\![w]\!][\![v_h]\!], \tag{6}$$

where ∇_h denotes the broken gradient, $\eta > 0$ is large enough to ensure coercivity and

$$\{\varphi\} \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}(\varphi_{|T_1} + \varphi_{|T_2}) & \text{for all } F \in \mathcal{F}_h^{T_1} \cap \mathcal{F}_h^{T_2}, \\ \varphi_{|T} & \text{for all } F \in \mathcal{F}_h^{T} \cap \mathcal{F}_h^{b}. \end{cases}$$

The implementation does not require cubatures. Indeed, using linearity, the terms in square brackets in (6) can be rewritten as $\sum_{F \in \mathcal{F}_h} |F|\{\nu\nabla_h u_h\}\cdot \mathrm{n}_F [\![v_h(\hat{x}_F)]\!]$ and $\sum_{F \in \mathcal{F}_h} |F|\{\nu\nabla_h v_h\}\cdot \mathrm{n}_F [\![u_h(\hat{x}_F)]\!]$ respectively, and the penalty term can be safely approximated using the middle point rule by $\sum_{F \in \mathcal{F}_h} \frac{\eta|F|}{h_F} [\![u_h(\widehat{x}_F)]\!][\![v_h(\hat{x}_F)]\!]$. As the discrete bilinear form a_h has been naturally extended to $V(h) \times V_h$, the usual dG analysis techniques can be applied.

We consider the following approximation of (2):

$$\text{Find} \quad u_h \in V_h \quad \text{s.t.} \quad a_h(u_h, v_h) = \int_\Omega f v_h \quad \text{for all } v_h \in V_h. \tag{7}$$

Problem (7) has *only one* unknown per cell, as the elimination of face unknowns has been carried out in the interpolation process. Such an operation can be performed in practice when, for a given $\mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}$, the dependence of each $I_F(\mathbf{V})$ on $\mathbf{V}$ is limited to a few cell values (which is the case, e.g., for the barycentric interpolator). On the other hand, when each $I_F(\mathbf{V})$ potentially depends on all the values $\{v_T\}_{T \in \mathcal{T}_h}$ (as it is the case for the interpolator obtained by expressing face unknowns in terms of cell unknowns in the hybrid methods of [5]) such an elimination need not be implemented.

2. Main results and comments

Throughout this section, $\{u_h\}_{h \in \mathcal{H}}$ will denote the sequence of solutions of the discrete problem (7) on the admissible mesh family $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$. Proceeding as in Arnold et al. [3], one can prove the following result:

Theorem 2 *(Convergence to regular solutions). Let $u \in H_0^1(\Omega)$ solve* (2) *and assume, moreover, that $u \in V$. Then, there exists C_u independent of the meshsize h such that*

$$\|u - u_h\|_h \leqslant C_u h.$$

Furthermore, under suitable additional regularity assumptions, the Aubin–Nitsche trick yields $\|u - u_h\|_{L^2(\Omega)} \leqslant C_u h^2$. The above estimates are numerically verified in Table 1. For all $v_h \in V_h$ and for all $F \in \mathcal{F}_h$, define the lifting operator solution of the following problem: Find $\mathrm{r}_F([\![v_h]\!]) \in [P_h^0]^d$ such that

$$\int_\Omega \mathrm{r}_F([\![v_h]\!])\cdot \tau_h = \int_F [\![v_h]\!]\{\tau_h\}\cdot \mathrm{n}_F \quad \forall \tau_h \in \big[P_h^0\big]^d,$$

and set $\mathrm{R}_h([\![v_h]\!]) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \mathrm{r}_F([\![v_h]\!])$. Then, proceeding as in [4, Theorem 2.2] one can prove that, for all sequences $\{v_h\}_{h \in \mathcal{H}}$ bounded in the $\|\cdot\|_h$ norm, as $h \to 0$, there is $v \in H_0^1(\Omega)$ such that $v_h \to v$ in $L^2(\Omega)$ and $\nabla_h v_h - \mathrm{R}_h([\![v_h]\!])$ weakly converges to ∇v in $[L^2(\Omega)]^d$. Following the guidelines of [4, Theorem 3.1] one can then prove that

Table 1
Numerical example for a pure diffusion problem on a triangular mesh family ($u = \sin(\pi x)\sin(\pi y)$, $\nu = \mathbb{1}_d$, $f = 2\pi^2 \sin(\pi x)\sin(\pi y)$).

h	$\|u - u_h\|_{L^2(\Omega)}$	Order	$\|u - u_h\|_h$	Order
$\frac{1}{4}$	2.8891e−02	–	1.7460e−01	–
$\frac{1}{8}$	6.9933e−03	2.05	8.1249e−02	1.10
$\frac{1}{16}$	1.7562e−03	1.99	3.9300e−02	1.05
$\frac{1}{32}$	4.4048e−04	2.00	1.9467e−02	1.01
$\frac{1}{64}$	1.1098e−04	1.99	9.6534e−03	1.01
$\frac{1}{128}$	2.7795e−05	2.00	4.8156e−03	1.00
$\frac{1}{256}$	6.9559e−06	2.00	2.4051e−03	1.00

Table 2
Numerical example for a diffusion–advection-reaction problem on a triangular mesh family ($\|\cdot\|_{\nu,\beta,h}$ is the associated energy norm, $u = \sin(\pi x)\sin(\pi y)$, $\nu = \mathbb{1}_d$, $\beta = (2\pi, 0)$, $\mu = 0$, $f = 2\pi^2(\sin(\pi x) + \cos(\pi x))\sin(\pi y)$).

h	$\|u - u_h\|_{L^2(\Omega)}$	Order	$\|u - u_h\|_{\nu,\beta,h}$	Order
$\frac{1}{4}$	3.0457e−02	–	1.7902e−01	–
$\frac{1}{8}$	7.4629e−03	2.03	8.3564e−02	1.10
$\frac{1}{16}$	1.8935e−03	1.98	4.0205e−02	1.06
$\frac{1}{32}$	4.7634e−04	1.99	1.9815e−02	1.02
$\frac{1}{64}$	1.1978e−04	1.99	9.7981e−03	1.02
$\frac{1}{128}$	2.9997e−05	1.99	4.8796e−03	1.00
$\frac{1}{256}$	7.4900e−06	2.00	2.4344e−03	1.00

Theorem 3 *(Convergence to minimal regularity solutions). Let u solve* (2)*. Then, as* $h \to 0$*,*

(i) $u_h \to u$ *in* $L^2(\Omega)$*,*
(ii) $\nabla_h u_h \to \nabla u$ *and* $\nabla_h u_h - \mathrm{R}_h([\![u_h]\!]) \to \nabla u$ *in* $[L^2(\Omega)]^d$*, and*
(iii) $\sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![u_h]\!]\|^2_{L^2(F)} \to 0$.

Some comments are of order. The derivation of the method outlined in Section 1 remains valid taking inspiration from any alternative dG method for problem (1) (cf. [3] for a review). Moreover, the ideas of Section 1 virtually extend to any problem for which a dG method has been devised. As an example, for the diffusion–advection-reaction problem

$$\nabla\cdot(-\nu\nabla u + \beta u) + \mu u = f \quad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega,$$

with $\beta \in C^1(\overline{\Omega})$ and $\mu \geqslant 0$ such that $\mu + \frac{1}{2}\nabla\cdot\beta > 0$, the following bilinear form yields an optimally converging approximation (cf. Table 2 for a numerical example):

$$b_h(w, v_h) \stackrel{\text{def}}{=} a_h(w, v_h) - \int_\Omega w\beta\cdot\nabla_h v_h + \sum_{F\in\mathcal{F}_h} \int_F w^{\uparrow} [\![v_h]\!] + \int_\Omega \mu w v_h,$$

where $w^{\uparrow} \stackrel{\text{def}}{=} (\beta\cdot \mathrm{n}_F)\{w\} + \frac{|\beta\cdot \mathrm{n}_F|}{2}[\![w]\!]$ on every $F \in \mathcal{F}_h^i$ and as $w^{\uparrow} \stackrel{\text{def}}{=} (\beta\cdot \mathrm{n})^{\oplus} w$ on every $F \in \mathcal{F}_h^b$. Finally, the compactness analysis of [4] also apply, so that valuable tools for the analysis of nonlinear problems are available.

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