C. R. Acad. Sci. Paris, Ser. I 348 (2010) 697–702

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical Problems in Mechanics/Partial Differential Equations

A fictitious domain model for the Stokes/Brinkman problem with jump embedded boundary conditions

Un modèle de domaine fictif pour le problème de Stokes/Brinkman avec des conditions de saut immergées

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ARTICLE INFO

Article history:
Received 5 October 2009
Accepted after revision 15 April 2010
Available online 20 May 2010

Presented by Philippe G. Ciarlet

ABSTRACT

We present and analyze a new fictitious domain model for the Brinkman or Stokes/Brinkman problems in order to handle general jump embedded boundary conditions (J.E.B.C.) on an immersed interface. Our model is based on algebraic transmission conditions combining the stress and velocity jumps on the interface Σ separating two subdomains: they are well chosen to get the coercivity of the operator. It is issued from a generalization to vector elliptic problems of a previous model stated for scalar problems with jump boundary conditions (Angot (2003, 2005) [2,3]). The proposed model is first proved to be well-posed in the whole fictitious domain and some sub-models are identified. A family of fictitious domain methods can be then derived within the same unified formulation which provides various interface or boundary conditions, e.g. a given stress of Neumann or Fourier type or a velocity Dirichlet condition. In particular, we prove the consistency of the given-traction E.B.C. method including the so-called *do nothing* outflow boundary condition.

RÉSUMÉ

Nous présentons l'analyse d'une nouvelle méthode de domaine fictif pour des problèmes de Brinkman ou de Stokes/Brinkman permettant de traîter des conditions de sauts (J.E.B.C.) immergées générales. Notre modèle est basé sur des conditions de transmission algébriques combinant les sauts des vecteurs contrainte et vitesse sur l'interface Σ séparant deux sous-domaines. Elles sont bien choisies de façon à guarantir la coercivité de l'opérateur et issues de la généralisation à des problèmes elliptiques vectoriels d'un modèle établi dans le cas scalaire (Angot (2003, 2005) [2,3]). On prouve tout d'abord que le modèle proposé est globalement bien posé dans tout le domaine fictif et on en identifie certains sous-modèles. Une classe de méthodes est ensuite proposée dans la même formulation unifiée qui permet d'obtenir des conditions aux limites variées, comme par exemple une contrainte donnée de type Neumann ou Fourier ou une vitesse imposée sur la frontière immergée. En particulier, nous prouvons la consistance de la méthode E.B.C. pour une condition de traction imposée qui inclue la condition usuelle de sortie ouverte de l'écoulement.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.04.022

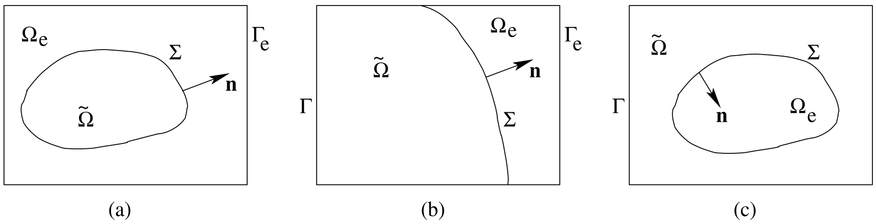

Fig. 1. Configurations for embedding the original domain $\tilde{\Omega}$ inside the fictitious domain $\Omega = \tilde{\Omega} \cup \Sigma \cup \Omega_e$.

Version française abrégée

Dans la Section 1, le problème aux limites originel de Brinkman (1)–(4) défini dans $\tilde{\Omega}$, avec une condition aux limites de Dirichlet sur le vecteur vitesse ou sur le vecteur pseudo-traction, est d'abord "immergé" dans le domaine fictif plus grand Ω, polygonal et de forme géométrique simple, voir Fig. 1. Cette extension (5)–(9) est une généralisation au présent cas vectoriel du modèle de fracture proposé dans [2] où un schéma numérique en volumes finis y est également décrit et analysé pour le résoudre. Les sauts de la vitesse $[\![\mathbf{u}]\!]_\Sigma$ et du vecteur contrainte $[\![\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}]\!]_\Sigma$ sur l'interface immergée Σ sont reliés par deux conditions de transmission algébriques (8)–(9). Le problème de domaine fictif (5)–(9) proposé est globalement bien posé dans l'espace $\mathbf{W} \times L^2(\Omega)$, cf. Théorème 1.1.

Les paramètres du modèle sur Σ ou dans Ω_e sont ensuite déterminés dans la Section 2 pour satisfaire exactement ou de façon approchée la condition aux limites immergée (19) de traction imposée par analogie avec la classe de méthodes proposée dans [3] pour un problème elliptique scalaire et numériquement validée dans [16,17]. Il apparaît que le choix simple $\mathbf{M} = 4\mathbf{S}$ permet de satisfaire (19) avec (18), indépendamment de tout contrôle extérieur dans Ω_e ou pénalisation de surface sur Σ. On montre dans le Théorème 2.1 que cette méthode (20) est consistante, i.e. $\mathbf{u}_{|\tilde{\Omega}} = \tilde{\mathbf{u}}$ et $p_{|\tilde{\Omega}} = \tilde{p}$ presque partout dans $\tilde{\Omega}$. D'autres variantes peuvent aussi être exhibées qui permettent à l'équation (17) dérivée de (8)–(9), d'approcher (19) par des techniques de pénalisation surfacique sur Σ ou volumique dans Ω_e de type H^1 voire L^2, cf. [1,5,9,3].

1. Fictitious domain model with embedded stress and velocity jumps on Σ

Notations. Let the domain $\Omega \subset \mathbb{R}^d$ (d=2 or 3 in practice) be an open bounded set, generally chosen convex and polygonal. Let an interface $\Sigma \subset \mathbb{R}^{d-1}$, Lipschitz continuous, separate Ω into two disjoint connected subdomains $\tilde{\Omega}$ and Ω_e such that $\Omega = \tilde{\Omega} \cup \Sigma \cup \Omega_e$. The boundaries of the domains are respectively defined by: $\partial\tilde{\Omega} = \Gamma \cup \Sigma$ for $\tilde{\Omega}$, $\partial\Omega_e = \Gamma_e \cup \Sigma$ for Ω_e and $\partial\Omega = \Gamma \cup \Gamma_e$ for Ω, see Fig. 1, assuming no cusp at $\Sigma \cap \partial\Omega$ in (b). Let $\mathbf{n}$ be the unit normal vector on Σ oriented from $\tilde{\Omega}$ to Ω_e. For a function ψ in $H^1(\tilde{\Omega} \cup \Omega_e)$, let ψ^- and ψ^+ be the traces of $\psi|_{\tilde{\Omega}}$ and $\psi|_{\Omega_e}$ on each side of Σ respectively, $\bar{\psi}_{|\Sigma} = (\psi^+ + \psi^-)/2$ the arithmetic mean of traces of ψ, and $[\![\psi]\!]_\Sigma = (\psi^+ - \psi^-)$ the jump of traces of ψ on Σ oriented by $\mathbf{n}$.

The proposed model is original compared to other fictitious domain methods, such as for instance [1,5,8,9,11,12,15] and the references therein. Our objective is to solve, with a fictitious domain method in Ω, the following generalized Brinkman problem, e.g. [1] and the references therein, originally defined in $\tilde{\Omega} \subset \Omega$ with either a boundary condition for the stress vector $\boldsymbol{\sigma}(\tilde{\mathbf{u}}, \tilde{p}) \cdot \mathbf{n} \equiv -\tilde{p}\mathbf{n} + \tilde{\mu}(\nabla\tilde{\mathbf{u}} + \nabla\tilde{\mathbf{u}}^t) \cdot \mathbf{n}$ or a Dirichlet boundary condition for the velocity $\tilde{\mathbf{u}}$ on Σ:

$$-\nabla \cdot \boldsymbol{\sigma}(\tilde{\mathbf{u}}, \tilde{p}) + \mu\mathbf{K}^{-1}\tilde{\mathbf{u}} = \mathbf{f} \quad \text{in } \tilde{\Omega}, \tag{1}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0 \quad \text{in } \tilde{\Omega}, \tag{2}$$

$$\tilde{\mathbf{u}} = 0 \quad \text{on } \Gamma, \tag{3}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_D \quad \text{with} \int_\Sigma \mathbf{u}_D \cdot \mathbf{n}\, ds = 0, \quad \text{or} \quad \boldsymbol{\sigma}(\tilde{\mathbf{u}}, \tilde{p}) \cdot \mathbf{n} + \mathbf{A}\tilde{\mathbf{u}} = \mathbf{q} \quad \text{on } \Sigma, \tag{4}$$

where $\mathbf{f} \in L^2(\tilde{\Omega})^d$, $\mathbf{u}_D \in H_{00}^{1/2}(\Sigma)^d$, $\mathbf{A} \in L^\infty(\Sigma)^{d\times d}$ a uniformly positive matrix and $\mathbf{q} \in H^{-\frac{1}{2}}(\Sigma)^d$ are given on Σ. The viscosity coefficients $\tilde{\mu}, \mu > 0$ and the permeability tensor $\mathbf{K}$, symmetric and uniformly positive definite, are bounded and given in $\tilde{\Omega}$. The Stokes problem is recovered with $\tilde{\mu} = \mu$ and $\mathbf{K}^{-1} = \varepsilon\,\mathbf{I}$ by taking the limit for $\varepsilon \to 0^+$, see [1].

This problem is thus "extended" to the whole fictitious domain Ω in the following manner by generalizing to vector elliptic problems the fracture model introduced in [2] for the scalar elliptic case, and where a finite volume numerical method was also proposed and analyzed.

For the data $\mathbf{f} \in L^2(\Omega)^d$, $\mathbf{g}$ and $\mathbf{h}$ given in $H^{-\frac{1}{2}}(\Sigma)^d$, we consider the elliptic problem including *immersed transmission conditions* on the interface Σ which link the trace jumps of both the stress vector $\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n} \equiv (-p\,\mathbf{I} + 2\tilde{\mu}\,\mathbf{d}(\mathbf{u})) \cdot \mathbf{n}$ with $\mathbf{d}(\mathbf{u}) \equiv \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ and the velocity vector $\mathbf{u}$ through Σ:

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) + \mu\,\mathbf{K}^{-1}\,\mathbf{u} = \mathbf{f} \quad \text{in } \tilde{\Omega} \cup \Omega_e, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \tilde{\Omega} \cup \Omega_e, \tag{6}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma \cup \Gamma_e, \tag{7}$$

$$\big[\!\big[\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}\big]\!\big]_\Sigma = \mathbf{M}\,\bar{\mathbf{u}}_{|\Sigma} - \mathbf{h} \quad \text{on } \Sigma, \tag{8}$$

$$\overline{\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}}_{|\Sigma} = \mathbf{S}\,[\![\mathbf{u}]\!]_\Sigma - \mathbf{g} \quad \text{on } \Sigma, \tag{9}$$

where the uniformly positive definite and symmetric permeability tensor $\mathbf{K} \equiv (K_{ij})_{1\leqslant i,j\leqslant d}$, the viscosity coefficients $\tilde{\mu}, \mu$, and the *transfer matrices* $\mathbf{S}, \mathbf{M}$ in the J.E.B.C. (8)–(9) on Σ are measurable and bounded functions verifying ellipticity assumptions:

$$\mathbf{K} \in \big(L^\infty(\Omega)\big)^{d\times d}; \qquad \exists K_0 > 0,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \mathbf{K}(x)^{-1} \cdot \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant K_0 |\boldsymbol{\xi}|^2 \quad \text{a.e. in } \Omega. \tag{$A1$}$$

$$\tilde{\mu}, \mu \in L^\infty(\Omega); \qquad \exists \mu_0 > 0, \quad \tilde{\mu}(x), \mu(x) \geqslant \mu_0 > 0 \quad \text{a.e. in } \Omega. \tag{$A2$}$$

$$\mathbf{M}, \mathbf{S} \in \big(L^\infty(\Sigma)\big)^{d\times d}; \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \mathbf{M}(x) \cdot \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0, \quad \mathbf{S}(x) \cdot \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0 \quad \text{a.e. on } \Sigma. \tag{$A3$}$$

The data $\mathbf{K}$, $\tilde{\mu}$, μ, $\mathbf{f}$ in Ω_e and $\mathbf{M}$, $\mathbf{S}$, $\mathbf{g}$, $\mathbf{h}$ on Σ will be chosen further in Section 2 in order to get: $\mathbf{u}|_{\tilde{\Omega}} = \tilde{\mathbf{u}}$ or at least $\mathbf{u}_\varepsilon|_{\tilde{\Omega}} \approx \tilde{\mathbf{u}}$ when the model is penalized with a penalty parameter $\varepsilon > 0$ such that $\varepsilon \to 0$.

With usual notations for Sobolev spaces, e.g. [14], we now define the Hilbert space:

$$\mathbf{W} \equiv \big\{\mathbf{v} \in L^2(\Omega)^d,\ \mathbf{v}_{|\tilde{\Omega}} \in H^1(\tilde{\Omega})^d \text{ and } \mathbf{v}_{|\Omega_e} \in H^1(\Omega_e)^d;\ \nabla \cdot \mathbf{v} = 0 \text{ in } \tilde{\Omega} \cup \Omega_e;\ \mathbf{v}_{|\Gamma\cup\Gamma_e} = 0 \text{ on } \Gamma \cup \Gamma_e\big\},$$

equipped with the natural inner product and associated norm in $H^1(\tilde{\Omega} \cup \Omega_e)^d$. With $\mathbf{u} \in \mathbf{W}$ satisfying (5) with $\mathbf{f} \in L^2(\Omega)^d$ such that $\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) \in L^2(\Omega)^d$, we can define $\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}^\pm_{|\Sigma}$ in $H^{-\frac{1}{2}}(\Sigma)^d$, see [13,6].

Then we prove the well-posedness of the problem (5)–(9) in Ω.

Theorem 1.1 *(Global solvability of the fictitious domain model with J.E.B.C.). If the ellipticity assumptions* $(A1)$–$(A3)$ *hold, the problem* (5)–(9) *with* $\mathbf{f} \in L^2(\Omega)^d$ *and* $\mathbf{g}, \mathbf{h} \in H^{-\frac{1}{2}}(\Sigma)^d$ *has a unique solution* $(\mathbf{u}, p) \in \mathbf{W} \times L^2(\Omega)$ *satisfying the weak form* (13) *for all* $\mathbf{v} \in \mathbf{W}$ *and such that* $p_{|\tilde{\Omega}} = p_0 + C_0 - C_1/2$ *and* $p_{|\Omega_e} = p_0 + C_0 + C_1/2$ *where* $p_0 \in L^2_0(\Omega) = \{q \in L^2(\Omega), \int_\Omega q\,\mathrm{d}x = 0\}$ *and* C_0, C_1 *are constants defined by*:

$$C_0 = \frac{1}{|\Sigma|}\big\langle \overline{\boldsymbol{\sigma}(\mathbf{u}, p_0) \cdot \mathbf{n}}_{|\Sigma} - \mathbf{S}\,[\![\mathbf{u}]\!]_\Sigma + \mathbf{g}, \mathbf{n}\big\rangle_{-\frac{1}{2},\Sigma} \quad \textit{and} \quad C_1 = \frac{1}{|\Sigma|}\big\langle \big[\!\big[\boldsymbol{\sigma}(\mathbf{u}, p_0) \cdot \mathbf{n}\big]\!\big]_\Sigma - \mathbf{M}\,\bar{\mathbf{u}}_{|\Sigma} + \mathbf{h}, \mathbf{n}\big\rangle_{-\frac{1}{2},\Sigma}.$$

It means that the pressure field $p \in L^2(\Omega)$ *must be adjusted from the zero-average pressure* $p_0 \in L^2_0(\Omega)$ *such that:* $\overline{(p - p_0)}_{|\Sigma} = C_0$ *and* $[\![p - p_0]\!]_\Sigma = C_1$ *to satisfy* (8)–(9) *in* $H^{-\frac{1}{2}}(\Sigma)^d$.

Moreover, there exists a constant $\alpha_0(\tilde{\Omega}, \Omega_e, K_0, \mu_0) > 0$ *such that*:

$$\|\mathbf{u}\|_{\mathbf{W}} + \|p_0\|_{0,\Omega} \leqslant \frac{c(\tilde{\Omega}, \Omega_e, \mu, \tilde{\mu}, \|\mathbf{K}^{-1}\|_\infty)}{\alpha_0}\big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{-\frac{1}{2},\Sigma} + \|\mathbf{h}\|_{-\frac{1}{2},\Sigma}\big).$$

Sketch of proof. We begin by deriving the weak form of the problem (5)–(9). With (5)–(7) and using the Green–Stokes formula, $\langle .\,, .\rangle_{-\frac{1}{2},\Sigma}$ being the duality pairing between $H^{-\frac{1}{2}}(\Sigma)^d$ and $H^{\frac{1}{2}}(\Sigma)^d$, we get respectively over $\tilde{\Omega}$ and Ω_e:

$$2\int_{\tilde{\Omega}} \tilde{\mu}\mathbf{d}(\mathbf{u}) : \mathbf{d}(\mathbf{v})\,\mathrm{d}x - \big\langle \boldsymbol{\sigma}(\mathbf{u}, p)^- \cdot \mathbf{n}, \mathbf{v}^-\big\rangle_{-\frac{1}{2},\Sigma} + \int_{\tilde{\Omega}} \mu\mathbf{K}^{-1}\,\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x = \int_{\tilde{\Omega}} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{W},\ \mathbf{v}_{|\Omega_e} = 0,$$

$$2\int_{\Omega_e} \tilde{\mu}\mathbf{d}(\mathbf{u}) : \mathbf{d}(\mathbf{v})\,\mathrm{d}x + \big\langle \boldsymbol{\sigma}(\mathbf{u}, p)^+ \cdot \mathbf{n}, \mathbf{v}^+\big\rangle_{-\frac{1}{2},\Sigma} + \int_{\Omega_e} \mu\mathbf{K}^{-1}\,\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x = \int_{\Omega_e} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{W},\ \mathbf{v}_{|\tilde{\Omega}} = 0.$$

Summing now the two previous equations yields:

$$2\int_{\tilde{\Omega}\cup\Omega_e} \tilde{\mu}\mathbf{d}(\mathbf{u}) : \mathbf{d}(\mathbf{v})\,\mathrm{d}x + \int_{\tilde{\Omega}\cup\Omega_e} \mu\mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x + \big\langle \boldsymbol{\sigma}(\mathbf{u}, p)^+ \cdot \mathbf{n}, \mathbf{v}^+\big\rangle_{-\frac{1}{2},\Sigma} - \big\langle \boldsymbol{\sigma}(\mathbf{u}, p)^- \cdot \mathbf{n}, \mathbf{v}^-\big\rangle_{-\frac{1}{2},\Sigma}$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{W}. \tag{10}$$

Then, noticing that for any bilinear form $\langle .\,, .\rangle_\Sigma$ defined on Σ we have the key equality below:

$$\big\langle U^+, V^+\big\rangle_\Sigma - \big\langle U^-, V^-\big\rangle_\Sigma = \big\langle [\![U]\!]_\Sigma, \overline{V}_{|\Sigma}\big\rangle_\Sigma + \big\langle \overline{U}_{|\Sigma}, [\![V]\!]_\Sigma\big\rangle_\Sigma, \quad \forall U, V \tag{11}$$

we obtain the following weak form in Ω:

$$2\int_{\tilde\Omega\cup\Omega_e}\tilde\mu\,\mathbf{d}(\mathbf{u}):\mathbf{d}(\mathbf{v})\,\mathrm{d}x+\int_{\tilde\Omega\cup\Omega_e}\mu\mathbf{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x+\big\langle[\![\boldsymbol\sigma(\mathbf{u},p)\cdot\mathbf{n}]\!]_\Sigma,\bar{\mathbf{v}}_{|\Sigma}\big\rangle_{-\frac12,\Sigma}+\big\langle\overline{\boldsymbol\sigma(\mathbf{u},p)\cdot\mathbf{n}}_{|\Sigma},[\![\mathbf{v}]\!]_\Sigma\big\rangle_{-\frac12,\Sigma}$$
$$=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x,\quad\forall\mathbf{v}\in\mathbf{W}.\tag{12}$$

Then, using the jump transmission conditions (8)–(9) on Σ we get the nice weak formulation below:

Find $\mathbf{u}\in\mathbf{W}$ *such that* $\forall\mathbf{v}\in\mathbf{W}$,

$$2\int_{\tilde\Omega\cup\Omega_e}\tilde\mu\,\mathbf{d}(\mathbf{u}):\mathbf{d}(\mathbf{v})\,\mathrm{d}x+\int_{\tilde\Omega\cup\Omega_e}\mu\mathbf{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x+\int_\Sigma\mathbf{M}\bar{\mathbf{u}}_{|\Sigma}\cdot\bar{\mathbf{v}}_{|\Sigma}\,\mathrm{d}s+\int_\Sigma\mathbf{S}[\![\mathbf{u}]\!]_\Sigma\cdot[\![\mathbf{v}]\!]_\Sigma\,\mathrm{d}s$$
$$=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x+\big\langle\mathbf{g},[\![\mathbf{v}]\!]_\Sigma\big\rangle_{-\frac12,\Sigma}+\langle\mathbf{h},\bar{\mathbf{v}}_{|\Sigma}\rangle_{-\frac12,\Sigma}.\tag{13}$$

With the ellipticity assumptions $(A1)$–$(A3)$, it is now easy to verify using the Korn inequality, e.g. [7], in $\tilde\Omega$, Ω_e and standard trace lemmas [13] that the left-hand side of (13) is a bilinear continuous and coercive form in $\mathbf{W}\times\mathbf{W}$, whereas the right-hand side is a linear continuous form in $\mathbf{W}$. Hence, by the Lax–Milgram theorem, e.g. [14], we have existence and uniqueness of the weak solution $\mathbf{u}$ in $\mathbf{W}$.

Moreover, using the version of the De Rham theorem presented in e.g. [18,6] and the Stokes formula with test functions $\mathbf{v}=\boldsymbol\varphi\in\mathcal{C}_c^\infty$ in (13) compactly supported either in $\tilde\Omega$ or Ω_e such that $\operatorname{div}\boldsymbol\varphi=0$ in $\tilde\Omega$ or Ω_e respectively, we get existence and uniqueness (for $\tilde\Omega$, Ω_e connected) of the pressure restrictions $p_{0|\tilde\Omega}$ and $p_{0|\Omega_e}$ in $L_0^2(\tilde\Omega)$ and $L_0^2(\Omega_e)$ respectively. This defines the pressure field $p_0=p_{0|\tilde\Omega}+p_{0|\Omega_e}$ in $L_0^2(\Omega)$ over the whole domain Ω such that the Brinkman equations (5), (6) hold a.e. in $\tilde\Omega\cup\Omega_e$. Then, the J.E.B.C. (8), (9) are recovered a.e. on Σ from (13) with test functions including the contributions on Σ: $\mathbf{v}\in\mathbf{W}$ such that $[\![\mathbf{v}]\!]_\Sigma=0$, i.e. $\mathbf{v}\in\mathbf{V}$, to get (8) with the additive constant pressure jump C_1 on Σ and $\mathbf{v}\in\mathbf{W}$ such that $\bar{\mathbf{v}}_{|\Sigma}=0$ to get (9) with the additive pressure constant C_0. This requires the introduction of *ad-hoc* divergence-free extensions to recover these conditions in $H^{-\frac12}(\Sigma)^d$, as in [6, Chap. III] for the Stokes/Neumann problem with a stress boundary condition. Reciprocally, we can verify with usual density arguments that the solution $\mathbf{u}\in\mathbf{W}$ of (13) and the recovered pressure field $p\in L^2(\Omega)$ also satisfy the Brinkman problem (5)–(6) almost everywhere in $\tilde\Omega\cup\Omega_e$.

Finally, the estimate of $\|\mathbf{u}\|_{\mathbf{W}}$ comes from standard energy estimates, whereas $\|p_0\|_{0,\Omega}$ is estimated since $p_0\in L_0^2(\Omega)$ by the Nečas theorem, see [18,7], with $\|\nabla p_0\|_{-1,\tilde\Omega}$, $\|\nabla p_0\|_{-1,\Omega_e}$ respectively calculated by the Brinkman equation (5) in $\tilde\Omega$ and Ω_e. We get with the continuous embedding $L^2\subset H^{-1}$:

$$\|p_0\|_{0,\Omega}\leqslant c(\tilde\Omega,\Omega_e)\big(\|\nabla p_0\|_{-1,\tilde\Omega}+\|\nabla p_0\|_{-1,\Omega_e}\big)\leqslant C\big(\|\mathbf{u}\|_{\mathbf{W}}+\|\mathbf{f}\|_{0,\Omega}\big).\qquad\square\tag{14}$$

Remark 1 *(J.E.B.C. for the elasticity problem).* A fictitious domain model with similar jump embedded boundary conditions for the stress tensor and displacement vector can be derived as well and proved to be well-posed for the linear elasticity problem in continuum mechanics using a similar stress formulation.

Remark 2 *(Generalization and sub-models).* The present fictitious domain model can be naturally generalized to the unsteady Stokes/Brinkman problem. The case of the nonlinear Navier–Stokes/Brinkman problem [9] will be the topic of a further work: the inertia terms require to modify the J.E.B.C. (8)–(9) to include the contribution of the kinetic energy. Different well-posed sub-models can be then derived from this general framework, e.g. the model with given stress and velocity jumps on Σ, see Theorem 1.2. The model with continuous stress and velocity was studied in [1]. Besides, our general framework allows to prove the global solvability of some models with physically relevant stress or velocity jump boundary conditions for the momentum transport at a fluid-porous interface, see [4].

The sub-model (5)–(7) with given stress and velocity jumps on Σ, i.e. $[\![\mathbf{u}]\!]_\Sigma=\boldsymbol\Phi\in H^{\frac12}(\Sigma)^d$ and $[\![\boldsymbol\sigma(\mathbf{u},p)\cdot\mathbf{n}]\!]_\Sigma=\boldsymbol\Psi\in H^{-\frac12}(\Sigma)^d$ is obtained from (5)–(9) by a surface penalty on Σ when $\varepsilon\to0$ with:

$$[\![\boldsymbol\sigma(\mathbf{u}_\varepsilon,p_\varepsilon)\cdot\mathbf{n}]\!]_\Sigma=\boldsymbol\Psi,\quad\text{with }\mathbf{M}=0,\ \mathbf{h}=-\boldsymbol\Psi\text{ on }\Sigma\tag{15}$$
$$\overline{\boldsymbol\sigma(\mathbf{u}_\varepsilon,p_\varepsilon)\cdot\mathbf{n}}_{|\Sigma}=\frac1\varepsilon\big([\![\mathbf{u}_\varepsilon]\!]_\Sigma-\boldsymbol\Phi\big),\quad\text{with }\mathbf{S}=\frac1\varepsilon\mathbf{I},\ \mathbf{g}=\frac1\varepsilon\boldsymbol\Phi\text{ on }\Sigma.\tag{16}$$

Indeed, by constructing a suitable extension to come back to $\boldsymbol\Phi=0$, we can prove Theorem 1.2 below. This sub-model was early considered and analyzed in [10] in the case of scalar transmission conditions for diffraction or scattering problems. This is also the basic model of the Immersed Interface Methods [11,12] for both scalar and vector problems.

Theorem 1.2 *(Convergence to sub-model with given stress and velocity jumps on Σ). For all $\varepsilon > 0$, let $(\mathbf{u}_\varepsilon, p_\varepsilon) \in \mathbf{W} \times L^2(\Omega)$ be the solution by Theorem* 1.1 *of the problem* (5)–(7), (15), (16) *with $\mathbf{f} \in L^2(\Omega)^d$, $\boldsymbol{\Psi} \in H^{-\frac{1}{2}}(\Sigma)^d$ and $\boldsymbol{\Phi} = 0$. Then, there exists a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times L^2(\Omega)$ of the problem* (5)–(7) *with $[\![\mathbf{u}]\!]_\Sigma = 0$ and $[\![\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}]\!]_\Sigma = \boldsymbol{\Psi}$ which is the limit of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when $\varepsilon \to 0$. Moreover, there exists a constant $C(\tilde{\Omega}, \Omega_e, \mu, \tilde{\mu}, \mathbf{K}) > 0$ and $\mathbf{Q} \in L^2(\Sigma)^d$ defined by: $\mathbf{Q} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\![\mathbf{u}_\varepsilon]\!]_\Sigma = \overline{\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}}_{|\Sigma}$, such that*:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}\|_W + \|p_{0\varepsilon} - p_0\|_{0,\Omega} \leqslant C\|\mathbf{Q}\|_{0,\Sigma}\sqrt{\varepsilon} \quad \textit{and} \quad \big\|[\![\mathbf{u}_\varepsilon]\!]_\Sigma\big\|_{0,\Sigma} \leqslant \|\mathbf{Q}\|_{0,\Sigma}\,\varepsilon.$$

If $\mathbf{Q} \in H^{\frac{1}{2}}(\Sigma)^d$, then an optimal error estimate in $\mathcal{O}(\varepsilon)$ holds.

2. Fictitious domain methods for the Stokes/Brinkman problem

In the previous fictitious domain model, the four unknown quantities $\boldsymbol{\sigma}(\mathbf{u}, p)^-_\Sigma \cdot \mathbf{n}$, $\boldsymbol{\sigma}(\mathbf{u}, p)^+_\Sigma \cdot \mathbf{n}$, $\mathbf{u}^-_\Sigma$ and $\mathbf{u}^+_\Sigma$ are linked with the algebraic transmission conditions (8), (9) on Σ. Hence, by eliminating one of the two exterior quantities $\boldsymbol{\sigma}(\mathbf{u}, p)^+_\Sigma \cdot \mathbf{n}$ or $\mathbf{u}^+_\Sigma$, let us say $\boldsymbol{\sigma}(\mathbf{u}, p)^+_\Sigma \cdot \mathbf{n}$ here, $\boldsymbol{\sigma}(\mathbf{u}, p)^-_\Sigma \cdot \mathbf{n}$ can be written as follows:

$$-\boldsymbol{\sigma}(\mathbf{u}, p)^-_\Sigma \cdot \mathbf{n} = \left(\mathbf{S} + \frac{1}{4}\mathbf{M}\right)\mathbf{u}^-_\Sigma - \left(\mathbf{S} - \frac{1}{4}\mathbf{M}\right)\mathbf{u}^+_\Sigma + \mathbf{g} - \frac{1}{2}\mathbf{h}, \quad \text{on } \Sigma. \tag{17}$$

Then, Eq. (17) appears to be similar to a Fourier type boundary condition on Σ, as in (4), for the restriction to $\tilde{\Omega}$ of the fictitious domain solution $(\mathbf{u}, p)_{|\tilde{\Omega}}$ if the exterior quantity $\mathbf{u}^+_\Sigma$ can be *controlled* by the fictitious domain problem restricted to the exterior domain Ω_e. For example, $\mathbf{u}^+_\Sigma$ can be enforced to tend to a given value, let us say zero, by H^1 or only L^2 volume penalty methods performed with the parameters $\tilde{\mu}$, μ, $\mathbf{K}$ and $\mathbf{f}$ properly chosen in Ω_e, see [1,3]. Moreover, the particular choice $\mathbf{M} = 4\mathbf{S}$ requires no *exterior control* since Eq. (17) yields the Fourier boundary condition below, independently of $\mathbf{u}^+_\Sigma$ or $\boldsymbol{\sigma}(\mathbf{u}, p)^+_\Sigma \cdot \mathbf{n}$:

$$-\boldsymbol{\sigma}(\mathbf{u}, p)^-_\Sigma \cdot \mathbf{n} = \frac{1}{2}\mathbf{M}\,\mathbf{u}^-_\Sigma + \mathbf{g} - \frac{1}{2}\mathbf{h}, \quad \text{on } \Sigma. \tag{18}$$

When the given-traction boundary condition in (4) is desired for the original problem (1)–(4) in $\tilde{\Omega}$, the following immersed boundary condition must be satisfied on Σ, or at least approximated by penalization methods for instance, by the solution of the fictitious domain problem in Ω, restricted to $\tilde{\Omega}$:

$$-\boldsymbol{\sigma}(\mathbf{u}, p)^-_\Sigma \cdot \mathbf{n} = \mathbf{A}\,\mathbf{u}^-_\Sigma - \mathbf{q}, \quad \text{on } \Sigma. \tag{19}$$

Let $0 < \varepsilon \ll 1$ be a real penalty parameter which is intended to tend to zero. Then several variants of the fictitious domain method are exhibited for the embedded traction boundary condition (19) on Σ. They are defined by giving sufficient conditions for the data such that Eq. (18) satisfy (19), or Eq. (17) be an approximation of (19) by surface penalty on Σ or volume penalty in Ω_e, as proposed for the scalar elliptic problem in [3]. These Jump Embedded Boundary Conditions (J.E.B.C.) methods are numerically experimented and validated for advection–diffusion problems in [17], or with diffuse interface in [16].

The variant with no exterior or surface control is then defined by:

$$\mathbf{M} = 4\mathbf{S} = 2\mathbf{A}, \quad \mathbf{h} - 2\mathbf{g} = 2\mathbf{q} \quad \text{on } \Sigma \quad \text{and} \quad \tilde{\mu}_{|\Omega_e} = 1, \quad \mu_{|\Omega_e} = 0, \quad \mathbf{f}_{|\Omega_e} = 0 \quad \text{in } \Omega_e. \tag{20}$$

It is the most natural choice but it involves both $[\![\mathbf{u}]\!]_\Sigma \neq 0$ and $[\![\boldsymbol{\sigma}(\mathbf{u}, p) \cdot \mathbf{n}]\!]_\Sigma \neq 0$, except for the so-called "do nothing" outflow boundary condition defined by $\boldsymbol{\sigma}(\tilde{\mathbf{u}}, \tilde{p}) \cdot \mathbf{n} = -p_e\,\mathbf{n}$ on Σ where $\mathbf{M} = \mathbf{S} = 0$.

We now prove the following proposition:

Theorem 2.1 *(Consistency of the given-traction E.B.C. method (20)). If the ellipticity assumptions* $(A1)$–$(A3)$ *hold, the problem* (5)–(9) *with $\mathbf{f}$ such that: $\mathbf{f}_{|\tilde{\Omega}} \in L^2(\tilde{\Omega})^d$, $\mathbf{f}_{|\Omega_e} = 0$ and $\mathbf{g}, \mathbf{h} \in H^{-\frac{1}{2}}(\Sigma)^d$ for the traction E.B.C. method* (20), *has a unique solution $(\mathbf{u}, p) \in \mathbf{W} \times L^2(\Omega)$ such that: $\mathbf{u}_{|\tilde{\Omega}} = \tilde{\mathbf{u}}$ and $p_{|\tilde{\Omega}} = \tilde{p}$ a.e. in $\tilde{\Omega}$. Here $(\tilde{\mathbf{u}}, \tilde{p})$ is the weak solution of the original traction problem* (1)–(4) *in $\tilde{\Omega}$ such that: $\tilde{\mathbf{u}} \in \mathbf{V}_N = \{\mathbf{v} \in H^1(\tilde{\Omega})^d;\ \nabla \cdot \mathbf{v} = 0, \mathbf{v}_{|\Gamma} = 0\}$ and $\tilde{p} = \tilde{p}_0 + \tilde{C}_0 \in L^2(\tilde{\Omega})$ with $\tilde{p}_0 \in L^2_0(\tilde{\Omega})$ and the constant $\tilde{C}_0$ defined by*:

$$\tilde{C}_0 = \frac{1}{|\Sigma|}\big\langle \boldsymbol{\sigma}(\tilde{\mathbf{u}}, \tilde{p}_0) \cdot \mathbf{n} + \mathbf{A}\,\tilde{\mathbf{u}} - \mathbf{q}, \mathbf{n}\big\rangle_{-\frac{1}{2},\Sigma}.$$

Sketch of proof. By defining the following bilinear form $a(.,.)$ and the linear form $l(.)$ in $\mathbf{V}_N$:

$$a(\tilde{\mathbf{u}}, \mathbf{v}) = 2\int_{\tilde{\Omega}} \tilde{\mu}\mathbf{d}(\tilde{\mathbf{u}}) : \mathbf{d}(\mathbf{v})\,\mathrm{d}x + \int_{\tilde{\Omega}} \mu\mathbf{K}^{-1}\,\tilde{\mathbf{u}} \cdot \mathbf{v}\,\mathrm{d}x + \int_{\Sigma} \mathbf{A}\,\tilde{\mathbf{u}} \cdot \mathbf{v}\,\mathrm{d}s, \qquad l(\mathbf{v}) = \int_{\tilde{\Omega}} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x + \langle \mathbf{q}, \mathbf{v}\rangle_{-\frac{1}{2},\Sigma} \tag{21}$$

and using the Lax–Milgram theorem, the given-traction problem (1)–(4) in $\tilde{\Omega}$ with the usual ellipticity assumptions has a unique weak solution $\tilde{\mathbf{u}} \in \mathbf{V}_N \equiv \{\mathbf{v} \in H^1_{0\Gamma}(\tilde{\Omega})^d, \nabla \cdot \mathbf{v} = 0\}$ such that:

$$a(\tilde{\mathbf{u}}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_N. \tag{22}$$

The pressure field $\tilde{p} = \tilde{p}_0 + \tilde{C}_0 \in L^2(\tilde{\Omega})$ can be now recovered with the De Rham theorem in order to satisfy the stress boundary condition (4) on Σ in the $H^{-\frac{1}{2}}$ sense, see [6] for the details. Moreover, the weak formulation of the fictitious domain problem (5)–(9) in Ω reads as (10), equivalently to (13). In particular, for a test function $\mathbf{v} \in \mathbf{W}$ such that $\mathbf{v}_{|\tilde{\Omega}} \in \mathbf{V}_N$ and $\mathbf{v}_{|\Omega_e} = 0$, we have:

$$2\int_{\tilde{\Omega}} \tilde{\mu}\mathbf{d}(\mathbf{u}) : \mathbf{d}(\mathbf{v})\,\mathrm{d}x + \int_{\tilde{\Omega}} \mu\mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x - \left\langle \boldsymbol{\sigma}(\mathbf{u}, p)^- \cdot \mathbf{n}, \mathbf{v}^- \right\rangle_{-\frac{1}{2},\Sigma} = \int_{\tilde{\Omega}} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{W},\ \mathbf{v}_{|\Omega_e} = 0. \tag{23}$$

For the E.B.C. method verifying (18) and thus (19) with the parameters given in (20), the fictitious domain solution $\mathbf{u}$ satisfies:

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) = a(\tilde{\mathbf{u}}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{W},\ \mathbf{v}_{|\Omega_e} = 0. \tag{24}$$

Hence, we get $a(\mathbf{u}_{|\tilde{\Omega}} - \tilde{\mathbf{u}}, \mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}_N$ which yields $\mathbf{u}_{|\tilde{\Omega}} = \tilde{\mathbf{u}}$ a.e. in $\tilde{\Omega}$.

Moreover, using the De Rham theorem in $\tilde{\Omega}$ as in the proof of Theorem 1.1, we get that $p_{0|\tilde{\Omega}} = \tilde{p}_0 \in L^2_0(\tilde{\Omega})$ a.e. in $\tilde{\Omega}$ (for $\tilde{\Omega}$ a connected open set). Finally, we verify with the parameters given in (20) that the pressure constants C_0, C_1 coming from the pressure recovering for the whole fictitious problem in Theorem 1.1 are such that $C_0 - C_1/2 = \tilde{C}_0$, which concludes the proof. □

Remark 3 *(Dirichlet E.B.C. methods).* Several variants of Dirichlet E.B.C. methods can be also exhibited by a straightforward generalization of the methods proposed in [3] and numerically experimented in [16,17]. Moreover, similar results of convergence can be then proved even if they are not precisely stated in this Note. In particular, the choice $\mathbf{A} = \frac{1}{\varepsilon}\mathbf{I}$ and $\mathbf{q} = \frac{1}{\varepsilon}\mathbf{u}_D$ in (19) allows to recover the Dirichlet condition in (4) with a surface penalty on Σ when $\varepsilon \to 0$.

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