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Probability Theory

Multidimensional BSDEs with super-linear growth coefficient: Application to degenerate systems of semilinear PDEs

EDSR multidimensionnelles à croissance surlinéaire : Application aux systèmes d'EDP dégénérées

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ABSTRACT

We establish the existence and uniqueness as well as the stability of p-integrable solutions to multidimensional backward stochastic differential equations (BSDEs) with super-linear growth coefficient and a p-integrable terminal condition ($p > 1$). The generator could neither be locally monotone in the variable y nor locally Lipschitz in the variable z. As application, we establish the existence and uniqueness of weak (Sobolev) solutions to the associated systems of semilinear parabolic PDEs. The uniform ellipticity of the diffusion matrix is not required. Our result covers, for instance, certain systems of PDEs with logarithmic nonlinearities which arise in physics.
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RÉSUMÉ

Nous établissons l'existence, l'unicité et la stability des solutions fortes pour des équations différentielles stochastiques rétrogrades (EDSR) avec une condition terminale p-integrable ($p > 1$) et un coefficient admettant des croissances surlinéaires en les deux variables y et z. De plus, ce dernier peut être ni locallement monotone en y ni localement Lipschitz en z. Nous montrons également l'existence et l'unicité des solutions faibles pour les systèmes d'EDP associés. L'uniforme ellipticité n'est pas requise pour la matrice de diffusion.
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Version française abrégée

La présente Note fait suite aux articles [1–3] sur les EDSR multidimensionnelles avec des conditions locales (en y et z) sur le coefficient. Elle étend ces derniers á des situations plus générales avec de nouvelles méthodes basées sur des estimations non standards, et en particulier, le lemme de Gronwall n'est pas utilisé. Nous établissons également l'existence et l'unicité de solutions faibles (Sobolev) pour des systémes d'EDP dégénérées. Les résultats essentiels sont :

Théorème 0.1. *Sous les conditions* (**H.0**)–(**H.4**)*, l'EDSR* (1) *admet une unique solution* (Y, Z) *qui satisfait l'estimation du Théorème* 3.1.

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Théorème 0.2. *Sous les Hypothèses* (**A.0**)–(**A.4**)*, l'EDP* ($\mathcal{P}^{(g,F)}$) *admet une unique solution faible au sens de la définition* (2.1)*. De plus on a pour tout* $t \in [0,T]$*,*

$$\big(u\big(s, X_s^{t,x}\big), \sigma^*\nabla u\big(s, X_s^{t,x}\big)\big) = \big(Y_s^{t,x}, Z_s^{t,x}\big) \quad p.p.\text{-}(s,x,\omega)$$

où $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ *désigne l'unique solution du système d'EDS–EDSR* (3)*.*

1. Introduction

Let $(W_t)_{0\leqslant t\leqslant T}$ be a r-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable d-dimensional random variable. Let f be an $\mathbb{R}^d$-valued function defined on $[0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times r}$ such that for every $(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times r}$, the map $(t,\omega)\to f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. The BSDE under consideration is,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad 0\leqslant t\leqslant T. \tag{1}$$

Since the work [9], one of the interesting subject on BSDEs theory is to study the existence and uniqueness of a strong (i.e. $\mathcal{F}_t^W$-adapted) solution to Eq. (1) under local conditions on the generator f. The difficulty encountered is essentially due to the fact that the control variable Z is known implicitly, by the Itô martingale representation theorem, as the integrand of a Brownian stochastic integral. Consequently the usual localization procedure (by stopping times) does not work. In other hand, the comparison methods (used for one-dimensional BSDEs) do not work in multidimensional case. The present Note is a natural development of [1–3] where some results on existence and uniqueness, as well as the stability of strong solutions are established for multidimensional BSDEs with local assumptions on the coefficient f in the two variables y and z. To begin, let ξ be a p-integrable (with $1<p<2$) and consider the following example of BSDE,

$$Y_t = \xi - \int_t^T Y_s\log|Y_s|\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad 0\leqslant t\leqslant T. \tag{2}$$

It is worth noting that in Eq. (2), the coefficient $f(y) := -y\log|y|$ is not locally monotone and hence not locally Lipschitz. Moreover, its growth is big power than y. In our knowledge, there is no results on multidimensional BSDEs which cover this interesting example.

The first main purpose of the present Note consists in establishing a result on the existence and uniqueness as well as the stability of strong solutions to the BSDE (1) which take in Eq. (2) as well as, other interesting examples which are, in our knowledge, not covered by the previous works. For instance, our result cover the following example (the terminal data ξ is merely p-integrable with $p>1$) where the coefficient f is neither locally monotone in y nor locally Lipschitz in z, and moreover, f can has a super-linear growth in its two variables y, z. Indeed, let $\varepsilon_0 \in\,]0,1[$. Let $g(y) := y\log\frac{|y|}{1+|y|}$ and $h\in\mathcal{C}(\mathbb{R}^{dr};\mathbb{R}_+)\cap\mathcal{C}^1(\mathbb{R}^{dr}\setminus\{0\};\mathbb{R}_+)$ be such that:

$$h(z) = \begin{cases} |z|\sqrt{-\log|z|} & \text{if } |z| < 1-\varepsilon_0, \\ |z|\sqrt{\log|z|} & \text{if } |z| > 1+\varepsilon_0. \end{cases}$$

The function $f(y,z) := -y\log|y| + g(y)h(z)$ satisfies our assumptions.

The assumption (**A.4**), which is local in y, z and also in ω, enables us to cover the following example of BSDE with stochastic monotone generators. Example: we assume that (ξ, f) satisfy (**H.0**)–(**H.3**) and

$$(\mathbf{H'.4}) \quad \begin{cases} \text{There exists } K'\in\mathbb{R}_+ \text{ and a positive process } C \text{ satisfying } \mathbb{E}\int_0^T e^{q'C_s}\,\mathrm{d}s < \infty \quad (\text{for some } q'>0) \\ \quad\text{and such that:} \\ \langle y-y', f(t,\omega,y,z) - f(t,\omega,y',z')\rangle \\ \quad \leqslant K'|y-y'|^2\{C_t(\omega) + |\log|y-y'||\} + K'|y-y'||z-z'|\sqrt{C_t(\omega) + |\log|z-z'||}. \end{cases}$$

The existence and uniqueness of solutions to BSDE (1) with p-integrable terminal data ξ (for $1<p<2$), have been studied in [7,6] under more restrictive conditions on the coefficient f. It should be noted that our result of the first part (BSDE part) cover those of [6,7] with different proofs. Our method is based on non-standard estimates and allows to treat simultaneously the existence and uniqueness as well as, the stability of solutions by using the same computations.

The second main purpose of the present Note consists to establish the existence and uniqueness of weak (Sobolev) solutions to the degenerate system of semilinear PDEs associated to BSDE (1), see e.g. [8] for semilinear PDEs. We develop a method which allows to prove the uniqueness of the PDE by means of the uniqueness of its associated BSDE. We first establish the existence and uniqueness of solutions which are representable by the solution of a BSDE, and next we prove that any solution is unique. To do this, we first establish that 0 is the unique solution to the homogeneous PDE then we prove an equivalence between the uniqueness for the non-homogeneous semilinear PDE and the uniqueness for its associated homogeneous linear PDE by using BSDEs techniques.

2. Definition and assumptions

Throughout this paper, $p > 1$ is an arbitrary fixed real number and all the considered processes are $(\mathcal{F}_t)$-predictable.

2.1. Definition

A solution to the BSDE (1) is an $(\mathcal{F}_t)$-adapted and $\mathbb{R}^{d+dr}$-valued process (Y, Z) such that

$$\mathbb{E}\left(\sup_{t\leqslant T}|Y_t|^p + \left(\int_0^T |Z_s|^2\,\mathrm{d}s\right)^{\frac{p}{2}} + \int_0^T \big|f(s, Y_s, Z_s)\big|\,\mathrm{d}s\right) < +\infty$$

and satisfies (1).

2.2. Assumptions

We assume that: There exist $M \in \mathbb{L}^0(\Omega; \mathbb{L}^1([0,T]; \mathbb{R}_+))$, $K \in \mathbb{L}^0(\Omega; \mathbb{L}^2([0,T]; \mathbb{R}_+))$ and $\gamma \in]0, \frac{1\wedge(p-1)}{2}[$, such that (with $\lambda_s := 2M_s + \frac{K_s^2}{2\gamma}$) we have,

(**H.0**) $\mathbb{E}(|\xi|^p e^{\frac{p}{2}\int_0^T \lambda_s\,\mathrm{d}s}) < \infty$.
(**H.1**) f is continuous in (y, z) for a.e. (t, ω).
(**H.2**) There exist η, $f^0 \in \mathbb{L}^0(\Omega \times [0,T]; \mathbb{R}_+)$ satisfying $\mathrm{Max}[\mathbb{E}(\int_0^T e^{\int_0^s \lambda_r\,\mathrm{d}r}\eta_s\,\mathrm{d}s)^{\frac{p}{2}},\ \mathbb{E}(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r\,\mathrm{d}r} f_s^0\,\mathrm{d}s)^p] < \infty$ and such that for every t, y, z,

$$\big\langle y, f(t, y, z)\big\rangle \leqslant \eta_t + f_t^0|y| + M_t|y|^2 + K_t|y||z|.$$

(**H.3**) There exist $q > 1$, $\bar{\eta} \in \mathbb{L}^q(\Omega \times [0,T]; \mathbb{R}_+)$, $\alpha \in]1, p[$ and $\alpha' \in]1, p \wedge 2[$ such that for every t, y, z,

$$\big|f(t, \omega, y, z)\big| \leqslant \bar{\eta}_t + |y|^\alpha + |z|^{\alpha'}.$$

(**H.4**) There exist $q' > 0$, $\nu \in \mathbb{L}^{q'}(\Omega \times [0,T]; \mathbb{R}_+)$ and $K' \in \mathbb{R}_+$ such that for every $N \in \mathbb{N}$ and every y, $y'z$, z' satisfying $|y|, |y'|, |z|, |z'| \leqslant N$

$$\begin{aligned}&\big\langle y - y', f(t, \omega, y, z) - f\big(t, \omega, y', z'\big)\big\rangle \mathbb{1}_{\{\nu_t(\omega)\leqslant N\}}\\ &\quad\leqslant K'\big|y - y'\big|^2 \log A_N + \sqrt{K' \log A_N}\big|y - y'\big|\big|z - z'\big| + K'(\log A_N)/A_N\end{aligned}$$

where (A_N) is a increasing sequence of real numbers which satisfy $A_N > 1$, $\lim_{N\to\infty} A_N = \infty$ and $A_N \leqslant N^\mu$ for some $\mu > 0$.

3. The main results

Theorem 3.1. *Assume that* (**H.0**)–(**H.4**) *hold. Then, the BSDE* (1) *has a unique solution* (Y, Z) *which satisfies,*

$$\begin{aligned}&\mathbb{E}\sup_t |Y_t|^p e^{\frac{p}{2}\int_0^t \lambda_s\,\mathrm{d}s} + \mathbb{E}\left[\int_0^T e^{\int_0^s \lambda_r\,\mathrm{d}r}|Z_s|^2\,\mathrm{d}s\right]^{\frac{p}{2}}\\ &\quad\leqslant C\left\{\mathbb{E}|\xi|^p e^{\frac{p}{2}\int_0^T \lambda_s\,\mathrm{d}s} + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r\,\mathrm{d}r}\eta_s\,\mathrm{d}s\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r\,\mathrm{d}r} f_s^0\,\mathrm{d}s\right)^p\right\}\end{aligned}$$

for some constant C *depending only on* p *and* γ.

Theorem 3.2. *Let* (ξ, f) *satisfies* (**H.0**)–(**H.4**) *and* $(\xi^n, f_n)_n$ *satisfies* (**H.0**)–(**H.3**) *uniformly on n. Assume moreover that*

(a) $\xi^n \to \xi$ *a.s. and* $\sup_n \mathbb{E}(|\xi_n|^p \exp(\frac{p}{2}\int_0^T \lambda_s\,\mathrm{d}s)) < \infty$
(b) *For every* $N \in \mathbb{N}^*$, $\lim_n \rho_N(f_n - f) = 0$ *a.e.*
(c) *for every* $n \in \mathbb{N}^*$, *the BSDE corresponding to the data* (ξ^n, f_n) *has a* (*not necessary unique*) *solution* (Y^n, Z^n) *which satisfies,* $\sup_n \mathbb{E}(\sup_{t\leqslant T}|Y_t^n|^p e^{\frac{p}{2}\int_0^T \lambda_s\,\mathrm{d}s}) < \infty$.

Then, there exists $(Y,Z)\in\mathbb{L}^p(\Omega;\mathcal{C}([0,T];\mathbb{R}^d))\times\mathbb{L}^p(\Omega;\mathbb{L}^2([0,T];\mathbb{R}^{dr}))$ *such that*

i)
$$\mathbb{E}\Big(\sup_t|Y_t|^p e^{\frac{p}{2}\int_0^t\lambda_s\,ds}\Big)+\mathbb{E}\Bigg[\int_0^T e^{\int_0^s\lambda_r\,dr}|Z_s|^2\,ds\Bigg]^{\frac{p}{2}}$$
$$\leqslant C^{p,\gamma}\Bigg\{\mathbb{E}\big(|\xi|^p e^{\frac{p}{2}\int_0^T\lambda_s\,ds}\big)+\mathbb{E}\Bigg(\int_0^T e^{\int_0^s\lambda_r\,dr}\eta_s\,ds\Bigg)^{\frac{p}{2}}+\mathbb{E}\Bigg(\int_0^T e^{\frac12\int_0^s\lambda_r\,dr}f_s^0\,ds\Bigg)^p\Bigg\},$$

ii) *for every* $p'<p$, $(Y^n,Z^n)\to(Y,Z)$ *strongly in* $\mathbb{L}^{p'}(\Omega;\mathcal{C}([0,T];\mathbb{R}^d))\times\mathbb{L}^{p'}(\Omega;\mathbb{L}^2([0,T];\mathbb{R}^{dr}))$,
iii) *for every* $\hat\beta<\frac{2}{\alpha'}\wedge\frac{p}{\alpha}\wedge\frac{p}{\alpha'}\wedge q$, $\lim_{n\to\infty}\mathbb{E}\int_0^T|f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|^{\hat\beta}\,ds=0$,
iv) (Y,Z) *is the unique solution to the BSDE* (1).

3.1. Application to partial differential equations

Let $\sigma:\mathbb{R}^k\mapsto\mathbb{R}^{kr}$, $b:\mathbb{R}^k\mapsto\mathbb{R}^k$, $g:\mathbb{R}^k\mapsto\mathbb{R}^k$, and $F:[0,T]\times\mathbb{R}^k\times\mathbb{R}^d\times\mathbb{R}^{dr}\mapsto\mathbb{R}^d$ be measurable functions. Consider the system of semilinear PDEs

$$\big(\mathcal{P}^{(g,F)}\big)\quad\begin{cases}\frac{\partial u(t,x)}{\partial t}+\mathcal{L}u(t,x)+F(t,x,u(t,x),\sigma^*\nabla u(t,x))=0, & t\in\,]0,T[,\ x\in\mathbb{R}^k,\\ u(T,x)=g(x), & x\in\mathbb{R}^k,\end{cases}$$

where $\mathcal{L}:=\frac12\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{ij}^2+\sum_i b_i\partial_i$.

We assume throughout this section that $\sigma\in\mathcal{C}_b^3(\mathbb{R}^k,\mathbb{R}^{kr})$, *and* $b\in\mathcal{C}_b^2(\mathbb{R}^k,\mathbb{R}^k)$.

The Markovian BSDE associated to the PDE $(\mathcal{P}^{(g,F)})$ is given by the system of SDE–BSDE,

$$\begin{cases}X_s^{t,x}=x+\int_t^s b(X_r^{t,x})\,dr+\int_t^s\sigma(X_r^{t,x})\,dW_r, & t\leqslant s\leqslant T,\\ Y_s^{t,x}=g(X_T^{t,x})+\int_s^T f(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,dr-\int_s^T Z_r^{t,x}\,dW_r.\end{cases}\tag{3}$$

We define,

$$\mathcal{H}^{1+}:=\bigcup_{\delta\geqslant0,\beta>1}\Bigg\{v\in\mathcal{C}\big([0,T];\mathbb{L}^\beta\big(\mathbb{R}^k,e^{-\delta|x|}\,dx;\mathbb{R}^d\big)\big):\ \int_0^T\int_{\mathbb{R}^k}\big|\sigma^*\nabla v(s,x)\big|^\beta e^{-\delta|x|}\,dx\,ds<\infty\Bigg\}.$$

Definition 3.1. A (weak) solution of $(\mathcal{P}^{(g,F)})$ is a function $u\in\mathcal{H}^{1+}$ such that for every $t\in[0,T]$ and $\varphi\in C_c^1([0,T]\times\mathbb{R}^d)$

$$\int_t^T\Big\langle u(s),\frac{\partial\varphi(s)}{\partial s}\Big\rangle ds+\big\langle u(t),\varphi(t)\big\rangle=\big\langle g,\varphi(T)\big\rangle+\int_t^T\big\langle F\big(s,.,u(s),\sigma^*\nabla u(s)\big),\varphi(s)\big\rangle ds+\int_t^T\big\langle Lu(s),\varphi(s)\big\rangle ds$$

where $\langle f(s),h(s)\rangle:=\int_{\mathbb{R}^k}f(s,x)h(s,x)\,dx$.

An integrating by part shows that, $\langle Lu(s),\varphi(s)\rangle=-\int_{\mathbb{R}^k}\frac12\langle\sigma^*\nabla u(s,x);\sigma^*\nabla\varphi(s,x)\rangle\,dx-\langle u(s),\mathrm{div}(\tilde b\varphi)(s)\rangle$ where $\tilde b_i:=b_i-\frac12\sum_j\partial_j(\sigma\sigma^*)_{ij}$.

3.2. Assumptions

Assume that there exist $\delta\geqslant0$ and $\overline{p}>1$ such that

(**A.0**) $g(x)\in\mathbb{L}^{\overline{p}}(\mathbb{R}^k,e^{-\delta|x|}\,dx;\mathbb{R}^d)$.
(**A.1**) $F(t,x,.,.)$ is continuous for a.e. (t,x).
(**A.2**) There are $\eta'\in\mathbb{L}^{\frac{\overline{p}}{2}\vee1}([0,T]\times\mathbb{R}^k,e^{-\delta|x|}\,dt\,dx;\mathbb{R}_+)$, $f^{0'}\in\mathbb{L}^{\overline{p}}([0,T]\times\mathbb{R}^k,e^{-\delta|x|}\,dt\,dx;\mathbb{R}_+))$, and $M,M'\in\mathbb{R}_+$ such that

$$\big\langle y,F(t,x,y,z)\big\rangle\leqslant\eta'(t,x)+f^{0'}(t,x)|y|+\big(M+M'|x|\big)|y|^2+\sqrt{M+M'|x|}|y||z|.$$

(**A.3**) There are $\overline{\eta}'\in\mathbb{L}^q([0,T]\times\mathbb{R}^k,e^{-\delta|x|}\,dt\,dx;\mathbb{R}_+)$ (for some $q>1$), $\alpha\in\,]1,\overline{p}[$ and $\alpha'\in\,]1,\overline{p}\wedge2[$ such that

$$\big|F(t,x,y,z)\big|\leqslant\overline{\eta}'(t,x)+|y|^\alpha+|z|^{\alpha'}.$$

(**A.4**) There are $K, r \in \mathbb{R}_+$ such that for every $N \in \mathbb{N}$ and every x, y, y', z, z' satisfying $e^{r|x|}, |y|, |y'|, |z|, |z'| \leqslant N$,

$$\langle y - y'; F(t,x,y,z) - F(t,x,y',z')\rangle \leqslant K\log N\left(\frac{1}{N} + |y-y'|^2\right) + \sqrt{K\log N}|y-y'||z-z'|.$$

Theorem 3.3. *Let $p \in]\alpha \vee \alpha', \bar{p}[$ if $M' > 0$ and $p = \bar{p}$ if $M' = 0$. Assume moreover that* (**A.0**)–(**A.4**) *hold. Then,*

1) *The PDE $(\mathcal{P}^{(g,F)})$ has a unique (weak) solution u on $[0,T]$.*
2) *For every $t \in [0,T]$, there exists $D_t \subset \mathbb{R}^k$ satisfying $\int_{\mathbb{R}^k} \mathbb{1}_{D_t^c}\,dx = 0$, $(D_t^c := \mathbb{R}^k \setminus D_t)$ such that for every $t \in [0,T]$ and every $x \in D_t$, the BSDE $(E^{(\xi^{t,x}, f^{t,x})})$ has a unique solution $(Y^{t,x}, Z^{t,x})$ on $[t,T]$, where $\xi^{t,x} := g(X_T^{t,x})$ and $f^{t,x}(s,y,z) := \mathbb{1}_{\{s>t\}}F(s, X_s^{t,x}, y, z)$.*
3) *For every $t \in [0,T]$, $(u(s,X_s^{t,x}), \sigma^*\nabla u(s, X_s^{t,x})) = (Y_s^{t,x}, Z_s^{t,x})$ a.e. (s,x,ω).*
4) *There exists a positive constant C depending only on δ, M, M', p, $\bar{p}$, $|\sigma|_\infty$, $|b|_\infty$ and T such that*

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^k}|u(t,x)|^p e^{-\delta'|x|}\,dx + \int_0^T\int_{\mathbb{R}^k}|\sigma^*\nabla u(t,x)|^{p\wedge 2}e^{-\delta'|x|}\,dt\,dx$$
$$\leqslant C\left(\mathbb{1}_{[M'\neq 0]} + \int_{\mathbb{R}^k}|g(x)|^{\bar{p}}\,dx + \int_{\mathbb{R}^k}\int_0^T \eta'(s,x)^{\frac{\bar{p}}{2}\vee 1}\,ds\,dx + \int_{\mathbb{R}^k}\int_0^T f^{0'}(s,x)^{\bar{p}}\,ds\,dx\right)$$

where $\delta' = \delta + \kappa' + \mathbb{1}_{[M'\neq 0]}$ and $\kappa' := \frac{p\bar{p}M'T}{(\bar{p}-p)}\sup(4, \frac{2p}{p-1})$.

4. Proofs

Let $(Y^i, Z^i)_{i=1,2}$ be two solutions of the BSDE (1) corresponding to the data $(\xi^i, f_i)_{i=1,2}$. Using Itô's formula and combining Höder's, Young's and Doob's inequalities, one can prove that if the assumption (**H.2**) holds then the quantity $\Theta_p^i := \mathbb{E}(\sup_t |Y_t^i|^p) + \mathbb{E}(\int_0^T |Z_s^i|^2\,ds)^{\frac{p}{2}}$ is finite and we have the following estimate,

Lemma 4.1. *Let $(\xi^i, f_i)_{i=1,2}$ satisfies* (**H.3**) *(with the same $\bar{\eta}, \alpha$ and α'). Then, there exists $\beta = \beta(p,q,\alpha,\alpha') \in]1, p\wedge 2[$ such that for every $\varepsilon > 0$ there is an integer $N_\varepsilon = N_\varepsilon(p,q,\alpha,\alpha',K',\mu,q',\varepsilon,(A_N)_N)$ such that for every function f satisfying* (**H.4**)

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|Y_t^1 - Y_t^2|^\beta\Big) + \mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2\,ds\right)^{\frac{\beta}{2}} \leqslant N_\varepsilon\left[\mathbb{E}(|\xi^1-\xi^2|^\beta) + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_1 - f)_s + \rho_{N_\varepsilon}(f_2-f)_s\,ds\right]$$
$$+ \varepsilon\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \bar{\eta}_s^q\,ds + \mathbb{E}\int_0^T v_s^{q'}\,ds\right].$$

The uniqueness of solutions follows now by putting $\xi^1 = \xi^2$ and $f_1 = f_2$ in the previous lemma. The existence of solutions is deduced a suitable approximation of (ξ, f) and an appropriate localization procedure which is close to those given in [1–3]. However, in contrast to [3], we do not use the L^2-weak compactness of the approximating sequence (Y^n, Z^n). Our method consists to directly prove, by using Lemma 4.1, that the sequence (Y^n, Z^n) strongly converges in some L^q space $(1 < q < 2)$ and, the limit satisfies the BSDE (1). The stability of solutions (Theorem 3.2) is deduced by using the same estimates. The results are first established for a small time duration, and next, for an arbitrarily prescribed time duration by using a continuation procedure.

To *prove the PDEs part*, we establish the following claims.

Claim 1. *There exists a unique weak solution u to the problem $(\mathcal{P}^{(g,F)})$ such that: for every $t \in [0,T]$,*

$$\big(u(s, X_s^{t,x}), \sigma^*\nabla u(s, X_s^{t,x})\big) = \big(Y_s^{t,x}, Z_s^{t,x}\big) \quad \text{a.e. } (s,x,\omega).$$

Claim 2. 0 *is the unique solution to the problem*

$$\big(\mathcal{P}^{(0,-\operatorname{div}(\tilde{b})(x)y)}\big) \quad \begin{cases} \frac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + \operatorname{div}(\tilde{b})(x)w(t,x) = 0, & t \in]0,T[,\ x \in \mathbb{R}^k, \\ w(T,x) = 0, & x \in \mathbb{R}^k \end{cases}$$

which satisfies for some $\beta > 1$

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^k}\left|w(t,x)\right|^{\beta}+\left|w(t,x)\right|\mathrm{d}x+\int_0^T\int_{\mathbb{R}^k}\left|\sigma^*\nabla w(t,x)\right|^{\beta}+\left|\sigma^*\nabla w(t,x)\right|\mathrm{d}t\,\mathrm{d}x<\infty. \tag{4}$$

Claim 3. 0 *is the unique solution to the PDE* $(\mathcal{P}^{(0,0)})$ *satisfying the inequality* (4)*.*

Claim 4. 0 *is the unique solution to the PDE* $(\mathcal{P}^{(0,0)})$*.*

Claim 5. $(\mathcal{P}^{(g,F)})$ *has a unique solution* iff 0 *is the unique solution to the PDE* $(\mathcal{P}^{(0,0)})$*.*

We shall prove Claim 5. By Claim 1, there exists a unique solution u to $(\mathcal{P}^{(g,F)})$ such that $(u(s,X_s^{t,x}),\sigma^*\nabla u(s,X_s^{t,x}))=(Y_s^{t,x},Z_s^{t,x})$. Let v be another solution to $(\mathcal{P}^{(g,F)})$ and put $\hat{F}(t,x):=F(s,x,u(s,x),\sigma^*\nabla u(s,x))-F(s,x,v(s,x),\sigma^*\nabla v(s,x))$. The function $w:=u-v$ is then a solution of the problem $(\mathcal{P}^{(0,\hat{F})})$. In other hand, since $(0,\hat{F})$ satisfies the assumptions (**A.0**)–(**A.4**), then step 1 ensures the existence of a unique solution $\hat{w}$ to the problem $(\mathcal{P}^{(0,\hat{F})})$ such that, $\hat{w}(s,X_s^{t,x})=\hat{Y}_s^{t,x}$ and $\sigma^*\nabla\hat{w}(s,X_s^{t,x})=\hat{Z}_s^{t,x}$, where $(\hat{Y}_s^{t,x},\hat{Z}_s^{t,x})$ is the unique solution of the BSDE $\hat{Y}_s^{t,x}=\int_s^T\hat{F}(r,X_r^{t,x})\,\mathrm{d}r-\int_s^T\hat{Z}_r^{t,x}\,\mathrm{d}W_r$. The uniqueness of $(\mathcal{P}^{(0,\hat{F})})$ (which follows from Claim 4) allows us to deduce that $v(s,X_s^{t,x})=Y_s^{t,x}-\hat{Y}_s^{t,x}$ and $\sigma^*\nabla v(s,X_s^{t,x})=Z_s^{t,x}-\hat{Z}_s^{t,x}$. The uniqueness of the BSDE $(g(X_T^{t,x}),F)$ shows that $v(t,X_s^{t,x})=u(t,X_s^{t,x})$. We get that $v(t,x)=v(t,x)$ a.e. by an equivalence of norms given in [5] or [4].

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