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A steady Navier–Stokes model for compressible fluid with partially strong solutions

Un modèle stationaire de Navier–Stokes pour fluide incompressible avec des solutions partiellement fortes

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ABSTRACT

In this Note, we prove the existence of a partially strong solution to the steady Navier–Stokes equations for viscous barotropic compressible fluids, in a bounded simply connected domain of $\mathbb{R}^3$ with the prescribed generalized impermeability conditions $\mathbf{curl}^k\,\mathbf{u}\cdot\mathbf{n}=0$, $k=0,1,2$ on the boundary. We call the solution "partially strong" because only the divergence-free part of the velocity field and the associated effective pressure have regularity typical for strong solution, while the density and the gradient part of the velocity have regularity typical for weak solution.

RÉSUMÉ

Dans cette Note, nous démontrons l'existence de solutions partiellement fortes pour les équations de Navier–Stokes stationnaires descriptives de fluides visqueux barotropiques compressibles, dans un domaine borné de $\mathbb{R}^3$, avec des conditions aux limites d'imperméabilité généralisée $\mathbf{curl}^k\,\mathbf{u}\cdot\mathbf{n}=0$, $k=0,1,2$. Nous utilisons le libellé « solution partiellement forte » pour dire que les propriétés de régularité de la partie à divergence nulle du champ des vitesses et de la pression effective associée sont typiques d'une solution forte, tandis que les propriétés de régularité de la densité et de la partie complémentaire du champ des vitesses (gradient d'un potentiel) sont typiques d'une solution faible.

Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^3$, simplement connexe et de frontière $\partial\Omega$ de classe $C^{2,1}$. Dans le cas stationnaire, l'écoulement d'un fluide visqueux barotropique compressible est usuellement décrit par les équations de Navier–Stokes

$$\mathrm{Div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla\mathcal{P}(\rho)=\mu\Delta\mathbf{u}-(2\mu+\lambda)\nabla\,\mathrm{div}\,\mathbf{u}+\rho\mathbf{f}+\mathbf{g},\qquad \mathrm{div}(\rho\mathbf{u})=0,$$

où la pression $\mathcal{P}(\rho)$ est régie par une loi d'état de la forme $\mathcal{P}(\rho)=\rho^\gamma$, $\gamma>3$, ρ étant la densité (inconnue), $\mathbf{u}=(u_1,u_2,u_3)$ le champ des vitesses (inconnu), $\mathbf{f}$ et $\mathbf{g}$ les forces externes (données) respectivement relatives à l'unité de masse, à l'unité de volume. μ et λ sont les coefficients usuels de viscosité (des constantes positives données).

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Nous considérons le problème aux limites (**P**) défini par ces équations auxquelles il convient d'ajouter

* les contraintes $\rho \geqslant 0$, $\int_\Omega \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = m$ ($m > 0$ masse totale donnée), et d'associer
* la traditionnelle condition de flux nul, $\mathbf{u}\cdot\mathbf{n} = 0$ sur $\partial\Omega$, et
* des conditions aux limites complémentaires, les conditions d'imperméabilité généralisée caractérisant les flux de vorticité (notre choix), $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n} = 0$, $\mathbf{curl}^2\,\mathbf{u}\cdot\mathbf{n} = 0$ sur $\partial\Omega$.

Une formulation équivalente du problème (**P**) est facile à écrire en utilisant une décomposition de Helmholtz–Weyl du champ des vitesses, $\mathbf{u} = \mathbf{v} + \nabla\varphi$ où $\mathbf{v}$ est sa part à divergence nulle avec $\mathbf{v}\cdot\mathbf{n} = 0$ sur $\partial\Omega$, et où φ résout le problème de Neumann $\Delta\varphi = \operatorname{div}\mathbf{u}$, $\partial_\mathbf{n}\varphi|_{\partial\Omega} = 0$. En définitive, on obtient un problème aux limites ($\widetilde{\mathbf{P}}$) pour quatre fonctions inconnues $\mathbf{v}, \varphi, \rho, \Pi$ (les équations sont en (5)–(8) dans la partie anglaise). Π est la pression associée à $\mathbf{v}$ dite « pression effective ».

Soit $k \in \mathbb{N}$, $k > m/|\Omega|$, la mesure de l'ensemble $\{\mathbf{x} \in \Omega\colon\ \rho(\mathbf{x}) \geqslant k\}$ contrôlée par des puissances négatives de k, telle que décrite par P.L. Lions, est une indication précieuse pour introduire un k-modèle tronqué du problème ($\widetilde{\mathbf{P}}$) en modifiant convenablement les termes non linéaires et la pression devenant $\mathcal{P}_k(\rho)$ (voir la partie anglaise). Naturellement le caractère hyperbolique de l'équation de continuité demeurant, nous procédons également à une régularisation elliptique. Il en résulte l'énoncé du problème tronqué-régularisé $(\widetilde{\mathbf{P}})_{k,\varepsilon}$ (les équations sont en (9)–(12) dans la partie anglaise). Le paramètre $\varepsilon > 0$ étant fixé, une formulation de type point fixe est appropriée pour établir l'existence d'une solution forte en appliquant le principe de Leray–Schauder. Nous suivons la démonstration présentée dans [6], adaptant et modifiant celle donnée dans [9], nous vérifions en outre que $0 \leqslant \rho_\varepsilon \leqslant k$ presque partout dans Ω.

Essentielle est ensuite la recherche d'estimations a priori basée sur le contrôle intermédiaire de $\|\mathcal{P}_k(\rho)\|_2$: nous établissons des estimations indépendantes de k et de ε de la forme,

$$\big(\mu\|\mathbf{curl}\,\mathbf{v}\|_2^2 + (2\mu+\lambda)\|\Delta\varphi\|_2^2\big) + c_1\varepsilon\big(\big\|\nabla\rho^{\gamma/2}\big\|_2^2 + \big\|\rho^\gamma\big\|_1\big) \leqslant c_2$$
$$\text{et}\quad \|\Pi\|_2^2 \leqslant c_2.$$

Nous établissons d'autre part les estimations suivantes, dépendant de k, pour $1 \leqslant q < +\infty$,

$$\|\mathbf{v}\|_{1,q} + \|\nabla\varphi\|_{1,q} + \|\Pi\|_q \leqslant c_1 k^{1+2\gamma/3}$$
$$\text{et}\quad \varepsilon\|\nabla\rho\|_2 \leqslant \varepsilon k m + c_1 k^{3+2\gamma/3}.$$

Nous avons noté c_1 diverses constantes dépendant de $|\Omega|, m, \mu, \lambda$ et γ, et c_2 diverses constantes dépendant en outre de $\|\mathbf{f}\|_{6\gamma/(5\gamma-3)}$ et $\|\mathbf{g}\|_{6/5}$. Et en supposant $\gamma > 3$ et k suffisamment grand, on peut montrer rigoureusement que $|\{\rho \geqslant k-1\}| \leqslant \varepsilon c_1 \frac{4k}{(k-2)^{\gamma+1}}$.

Dans la formulation intégrale du problème $(\widetilde{\mathbf{P}})_{k,\varepsilon}$ à k fixé, les procédures de passage à la limite quand $\varepsilon \to 0$ sont bien connues, elles nécessitent une grande attention pour le terme contenant $\mathcal{P}(\rho)$, alors les principaux résultats sont, le deuxième apportant la régularité annoncée,

Théorème 1. *Il existe une solution faible* $(\mathbf{v}, \varphi, \rho, \Pi)$ *au problème* ($\widetilde{\mathbf{P}}$), $(\mathbf{v}, \varphi, \rho, \Pi) \in \mathbf{D}_\sigma^{1,q}(\Omega) \times W^{2,q}(\Omega) \times L^\infty(\Omega) \times L^q(\Omega)$ *pour tout* $q \in [1, +\infty)$. *En outre* ρ *satisfait l'inégalité* $\rho \leqslant k-1$ *presque partout dans* Ω *où* k *est suffisamment grand pour vérifier* $k > m/|\Omega|$ *et* $k^{1-\gamma/3} < c_1(1-\frac{2}{k})^{\gamma+1}$. *Les données sont* $m > 0$, $\gamma > 3$, $\mu > 0$, $2\mu + \lambda > 0$, $\mathbf{f}$ *et* $\mathbf{g}$ *dans* $\mathbf{L}^r(\Omega)$, $3 < r < +\infty$.

Théorème 2. *Toute solution faible* $(\mathbf{v}, \varphi, \rho, \Pi)$ *fournie par le théorème* 1 *possède les propriétés* $\mathbf{v} \in \mathbf{D}_\sigma^{2,r}(\Omega)$ *et* $\Pi \in W^{1,r}(\Omega)$. *Sous les hypothèses du théorème* 1, $(\rho, \mathbf{u} = \mathbf{v} + \nabla\varphi)$ *résout le problème* (**P**) *dans* $L^\infty(\Omega) \times \mathbf{D}_\sigma^{2,r}(\Omega) + \nabla W^{2,q}(\Omega)$.

On trouvera dans la partie anglaise les grandes lignes des démonstrations (tous les détails dans [7]). Cette Note se terminera par deux commentaires sur le choix des conditions aux limites et sur l'article [2].

1. Formulation of the problem

Notation. We assume that Ω is a bounded simply connected domain in $\mathbb{R}^3$ with the boundary of class $C^{2,1}$. We denote vector functions and spaces of vector functions by boldface letters. Thus, the corresponding Lebesgue spaces (respectively Sobolev spaces) are $\mathbf{L}^r(\Omega)$ (respectively $\mathbf{W}^{l,r}(\Omega)$). The norms are denoted $\|.\|_r$ (respectively $\|.\|_{l,r}$). The closure of $\{\mathbf{u} \in \mathbf{C}^\infty;\ \operatorname{div}\mathbf{u} = 0\}$ in $\mathbf{L}^r(\Omega)$ is denoted by $\mathbf{L}_\sigma^r(\Omega)$. Note that the functions $\mathbf{u} \in \mathbf{L}_\sigma^2(\Omega)$ satisfy $\mathbf{u}\cdot\mathbf{n}|_\Omega = 0$ in the sense of traces. We put $\mathbf{W}_\sigma^{l,r}(\Omega) := \mathbf{W}^{l,r}(\Omega) \cap \mathbf{L}_\sigma^r(\Omega)$ and we denote by $\mathbf{D}_\sigma^{1,r}(\Omega)$, the subspace of $\mathbf{W}_\sigma^{1,r}(\Omega)$ of functions, satisfying the additional boundary condition $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$. We know from [1] that the restriction of $\mathbf{curl}$ to the space $\mathbf{D}_\sigma^{1,2}(\Omega)$ is a self-adjoint operator in $\mathbf{L}_\sigma^2(\Omega)$ and $\mathbf{D}_\sigma^{1,2}(\Omega)$ can be characterized as $\{\mathbf{u} = \mathbf{u}_0 + \nabla\psi;\ \mathbf{u}_0 \in \mathbf{W}_0^{1,2}(\Omega),\ -\Delta\psi = \operatorname{div}\mathbf{u}_0 \text{ in } \Omega,\ \partial_\mathbf{n}\psi|_{\partial\Omega} = 0\}$.

We denote by (NS) the steady Navier–Stokes boundary value problem

$$\mathrm{Div}[\rho\mathbf{u}\otimes\mathbf{u}]+\nabla\mathcal{P}(\rho)=-\mu\,\mathbf{curl}^2\,\mathbf{u}+(2\mu+\lambda)\nabla\,\mathrm{div}\,\mathbf{u}+\rho\mathbf{f}+\mathbf{g}\quad\text{in }\Omega,\tag{1}$$

$$\mathrm{div}(\rho\mathbf{u})=0\quad\text{in }\Omega,\tag{2}$$

$$\rho\geqslant 0,\qquad\int_\Omega\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}=m,\tag{3}$$

$$\text{(i) }\mathbf{u}\cdot\mathbf{n}=0,\qquad\text{(ii) }\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}=0,\qquad\text{(iii) }\mathbf{curl}^2\,\mathbf{u}\cdot\mathbf{n}=0\quad\text{on }\partial\Omega.\tag{4}$$

We assume that $m>0$ is given and $\mathcal{P}(\rho)=\rho^\gamma$ with $\gamma>3$. These values of γ enable us to obtain the density ρ essentially bounded in Ω. Hereafter we put, for simplicity, $\mathbf{g}=0$.

Using the Helmholtz–Weyl decomposition $\mathbf{u}=\mathbf{v}+\nabla\varphi$, we obtain an equivalent formulation of problem (NS):

$$\mathrm{Div}\big[\rho(\mathbf{v}+\nabla\varphi)\otimes(\mathbf{v}+\nabla\varphi)\big]+\nabla\Pi=-\mu\,\mathbf{curl}^2\,\mathbf{v}+\rho\mathbf{f}\quad\text{in }\Omega,\tag{5}$$

$$\mathrm{div}\,\mathbf{v}=0\quad\text{in }\Omega,\tag{6}$$

$$-\Delta\varphi=\frac{1}{2\mu+\lambda}\big[\Pi-\mathcal{P}(\rho)\big]\quad\text{in }\Omega,\tag{7}$$

$$\mathrm{div}\big[\rho(\mathbf{v}+\nabla\varphi)\big]=0\quad\text{in }\Omega\tag{8}$$

with the constraints (3), the boundary condition $\partial_{\mathbf{n}}\varphi|_{\partial\Omega}=0$ and the boundary conditions (4) for $\mathbf{v}$, which we further refer to as to $(4)_{\mathbf{v}}$.

2. A regularized truncated problem

Let $k\in\mathbb{N}$, $k>1$. It is explained by P.L. Lions in [3] that the measure of the set $\{\mathbf{x}\in\Omega;\ \rho(\mathbf{x})\geqslant k\}$ can be controlled by means of negative powers of k. This is a motivation for truncating the density ρ and the pressure for $\rho\geqslant k$ or $\rho\geqslant k-1$ (see hereafter): we introduce a non-increasing smooth function F_k from $\mathbb{R}$ to $[0,1]$ such that $F_k(t)=0$ for $t<0$ or $t>k$, $F_k(t)=1$ for $0\leqslant t\leqslant k-1$ and $0\leqslant F_k(t)\leqslant 1$ for $k-1\leqslant t\leqslant k$. We define $\mathcal{P}_k$ and $\mathcal{N}$ by the equations

$$\mathcal{P}_k(\rho):=\int_0^\rho F_k(s)\mathcal{P}'(s)\,\mathrm{d}s,\qquad\mathcal{N}\big[F_k(\rho)\rho,\mathbf{u}\big]:=\frac12\,\mathrm{Div}\big[F_k(\rho)\rho\mathbf{u}\otimes\mathbf{u}\big]+\frac12 F_k(\rho)\rho\mathbf{u}\cdot\nabla\mathbf{u}.$$

One can easily verify that if $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0$ then $\int_\Omega\mathcal{N}[F_k(\rho)\rho,\mathbf{u}]\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}=0$, and if ρ and $\mathbf{u}$ are sufficiently smooth then $\mathcal{N}[F_k(\rho)\rho,\mathbf{u}]$ coincides with $\mathrm{Div}[\rho\mathbf{u}\otimes\mathbf{u}]$ on the set $\{\mathbf{x}\in\Omega;\ \rho(\mathbf{x})<k-1\}$.

The regularized truncated version of problem (NS), denoted by $(\mathrm{NS})_{k,\epsilon}$, consists of the constraints (3), the boundary conditions $(4)_{\mathbf{v}}$ and the equations (respectively the boundary conditions)

$$\mu\,\mathbf{curl}^2\,\mathbf{v}+\nabla\Pi=-\mathcal{N}\big[F_k(\rho)\rho,\mathbf{v}+\nabla\varphi\big]+F_k(\rho)\rho\mathbf{f}\quad\text{in }\Omega,\tag{9}$$

$$\mathrm{div}\,\mathbf{v}=0\quad\text{in }\Omega,\tag{10}$$

$$-\Delta\varphi=\frac{1}{2\mu+\lambda}\big[\Pi-\mathcal{P}(\rho)\big]\quad\text{in }\Omega,\tag{11}$$

$$-\epsilon\,\Delta\rho=\epsilon\left(\frac{m}{|\Omega|}-\rho\right)-\mathrm{div}\big[F_k(\rho)\rho(\mathbf{v}+\nabla\varphi)\big]\quad\text{in }\Omega,\tag{12}$$

$$\partial_{\mathbf{n}}\varphi=0\quad\text{on }\partial\Omega,\tag{13}$$

$$\partial_{\mathbf{n}}\rho=0\quad\text{on }\partial\Omega.\tag{14}$$

Function $\mathbf{f}\in\mathbf{L}^r(\Omega)$ (with $3<r<+\infty$) is supposed to be given and parameter ϵ is supposed to be positive. Using the advantage of the elliptic regularization in (12), one can show that if $k>m/|\Omega|$ then to given $(\mathbf{v},\varphi)\in\mathfrak{N}:=[\mathbf{W}^{1,\infty}(\Omega)\cap\mathbf{D}^{1,2}_\sigma(\Omega)]\times W^{2,\infty}(\Omega)$, there exists $\rho=\rho(\mathbf{v},\varphi)\in W^{2,p}(\Omega)$ for all $1\leqslant p<+\infty$, a solution of (12), (14), such that $0\leqslant\rho\leqslant k$ a.e. in Ω, satisfying (3).

Further, given $(\mathbf{w},\psi)\in\mathfrak{N}$, we solve the Stokes problem, i.e. Eq. (9) with the right hand side $-\mathcal{N}[F_k(\rho)\rho,\mathbf{w}+\nabla\psi]+F_k(\rho)\rho\mathbf{f}$ (where $\rho=\rho(\mathbf{w},\psi)$) with Eq. (10) and the boundary conditions $(4)_{\mathbf{v}}$. We find a solution $(\mathbf{v},\Pi)$ in $\mathbf{D}^{2,r}_\sigma(\Omega)\times W^{1,r}(\Omega)$. Afterwards, we solve the Neumann problem (11), (13) (with $\rho=\rho(\mathbf{w},\psi)$) and we find $\varphi\in W^{3,r}(\Omega)$. We denote $(\mathbf{v},\varphi)=\mathcal{T}(\mathbf{w},\psi)$. Finally, applying the Leray–Schauder theorem, we prove that operator $\mathcal{T}$ has a fixed point in $\mathfrak{N}$. Thus, we obtain a solution $(\mathbf{v},\varphi,\rho,\Pi)$ of problem $(\mathrm{NS})_{k,\epsilon}$ in $\mathbf{D}^{2,r}_\sigma(\Omega)\times W^{3,r}(\Omega)\times W^{2,p}(\Omega)\times W^{1,r}(\Omega)$.

3. Technical estimates of a solution of the problem $(\mathrm{NS})_{k,\epsilon}$

Here we explain in three steps how one can derive some important estimates.

Step 1. Multiplying Eq. (9) by $\mathbf{v}+\nabla\varphi$, integrating on Ω and expressing $\nabla\Pi$ from Eq. (11), we obtain

$$\mu\|\operatorname{\mathbf{curl}}\mathbf{v}\|_2^2+(2\mu+\lambda)\|\Delta\varphi\|_2^2+\int_\Omega \nabla\mathcal{P}_k(\rho)\cdot(\mathbf{v}+\nabla\varphi)\,\mathrm{d}\mathbf{x}=\int_\Omega F_k(\rho)\rho\mathbf{f}\cdot(\mathbf{v}+\nabla\varphi)\,\mathrm{d}\mathbf{x}, \tag{15}$$

where the integral on the left hand side can be rewritten, using successively (11), (10), $(4)_{\mathbf{v}}$, (12) and (10), as follows:

$$\int_\Omega \nabla\mathcal{P}_k(\rho)\cdot(\mathbf{v}+\nabla\varphi)\,\mathrm{d}\mathbf{x}=\frac{4\varepsilon}{\gamma}\int_\Omega\left|\nabla\rho^{\gamma/2}\right|^2\mathrm{d}\mathbf{x}+\frac{\varepsilon}{\gamma-1}\int_\Omega\rho^\gamma\,\mathrm{d}\mathbf{x}-\frac{\varepsilon\gamma}{\gamma-1}\int_\Omega\frac{m}{|\Omega|}\rho^{\gamma-1}\,\mathrm{d}\mathbf{x}.$$

The right hand side of (15) can be estimated so that we obtain the inequality

$$\begin{aligned}&\mu\|\operatorname{\mathbf{curl}}\mathbf{v}\|_2^2+(2\mu+\lambda)\|\Delta\varphi\|_2^2+\frac{4\varepsilon}{\gamma}\left\|\nabla\varphi^{\gamma/2}\right\|_2^2+\frac{2\varepsilon}{\gamma-1}\left\|\rho^\gamma\right\|_1\\&\quad\leqslant C\left\|\mathcal{P}_k(\rho)\right\|_2^{2/\gamma}\|\mathbf{f}\|^2_{6\gamma/(5\gamma-3)}+\frac{2\varepsilon}{\gamma-1}\frac{m^\gamma}{|\Omega|^{\gamma-1}},\end{aligned}$$

where C is independent of k and ϵ.

Step 2. Let M_Ω assign to each function from $L^1(\Omega)$ its mean value in Ω, and let $\mathfrak{B}$ be the so-called Bogovskij operator, mapping $\{\mathbf{f}\in\mathbf{L}^p(\Omega);\ M_\Omega(\mathbf{f})=0\}$ onto $\mathbf{W}_0^{1,p}(\Omega)$, with the property $\operatorname{div}\mathfrak{B}[\mathbf{f}]=\mathbf{f}$ a.e. in Ω. Integrating the product of Eq. (9) with $\boldsymbol{\phi}:=\mathfrak{B}[\mathcal{P}_k(\rho)-M_\Omega(\mathcal{P}_k(\rho))]$ in Ω, we obtain

$$\mu\int_\Omega\operatorname{\mathbf{curl}}^2\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}+\int_\Omega\mathcal{N}\left[F_k(\rho)\rho,\mathbf{v}+\nabla\varphi\right]\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}=-\int_\Omega\nabla\Pi\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}+\int_\Omega F_k(\rho)\rho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}, \tag{16}$$

where $\Pi=\mathcal{P}_k(\rho)-(2\mu+\lambda)\Delta\varphi$. Using the fact that $|\int_\Omega\mathcal{P}_k(\rho)M_\Omega(\mathcal{P}_k(\rho))|\leqslant\delta\|\mathcal{P}_k(\rho)\|_2^2+C(\delta)$, with an arbitrarily small parameter δ and a constant $C(\delta)$, also depending on $|\Omega|$, m, μ, λ and γ, one can verify that the first integral on the right hand side of (16) satisfies

$$-\int_\Omega\nabla\Pi\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}=\int_\Omega\Pi\operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\geqslant(1-\delta)\left\|P_k(\rho)\right\|_2^2-C(\delta)-C(\delta)\|\Delta\varphi\|_2^2.$$

All other integrals in (16) can be estimated so that we finally obtain the inequality

$$\left\|\mathcal{P}_k(\rho)\right\|_2^{2/\gamma}\|\mathbf{f}\|^2_{6\gamma/(5\gamma-3)}\leqslant\delta\big(\mu\|\operatorname{\mathbf{curl}}\mathbf{v}\|_2^2+(2\mu+\lambda)\|\Delta\varphi\|_2^2\big)+C(\delta)\|\mathbf{f}\|^{2\gamma/(\gamma-1)}_{6\gamma/(5\gamma-3)}+C. \tag{17}$$

Step 3. Multiplying Eq. (12) by $-(2\mu+\lambda)F_k(\rho)$, integrating on Ω, using the non-positiveness of F_k' and applying also successively (10), $(4)_{\mathbf{v}(i)}$ and (11), we derive the inequality

$$\begin{aligned}(2\mu+\lambda)\epsilon\int_\Omega\left(\rho-\frac{m}{|\Omega|}\right)F_k(\rho)\,\mathrm{d}\mathbf{x}&\geqslant(2\mu+\lambda)\int_\Omega\rho(\mathbf{v}+\nabla\varphi)F_k(\rho)F_k'(\rho)\nabla\rho\,\mathrm{d}\mathbf{x}\\&=\int_{k-1\leqslant\rho}\left(\int_{k-1}^{\rho}\frac{t}{2}\left[F_k^2(t)\right]'\mathrm{d}t\right)\big(\Pi-\mathcal{P}_k(\rho)\big)\,\mathrm{d}\mathbf{x}.\end{aligned}$$

The inside integral on the right hand side can be further estimated from below by $-\frac{1}{2}\rho$ and from above by $-\frac{1}{2}(k-2)$. Thus, one can derive the inequality

$$\frac{1}{2}(k-2)^{\gamma+1}\left|\{\rho\geqslant k-1\}\right|\leqslant(2\mu+\lambda)\epsilon\left(k-\frac{m}{|\Omega|}\right)|\Omega|+\int_{k-1\leqslant\rho}\frac{1}{2}\rho|\Pi|\,\mathrm{d}\mathbf{x}. \tag{18}$$

4. Existence of a weak solution

Combining (16) with (17), we can successively estimate $\mu\|\mathbf{curl}\,\mathbf{v}\|_2^2 + (2\mu+\lambda)\|\Delta\varphi\|_2^2$, $\|\mathcal{P}_k(\rho)\|_2$ and $\|\Pi\|_2$ by constants, independent of ϵ. On the other hand we can also derive estimates of $\mathbf{v}, \varphi, \rho, \Pi$ dependent on k. Precisely, we get for each $1 \leqslant q < +\infty$ $\|\mathbf{v}\|_{1,q} \leqslant Ck^{1+2\gamma/3}$, $\|\nabla\varphi\|_{1,q} \leqslant Ck^{1+2\gamma/3}$ and $\|\Pi\|_q \leqslant Ck^{1+2\gamma/3}$. Here, constant C is independent of q, ϵ and k. Using (18), we arrive at the estimate $|\{\rho \geqslant k-1\}| \leqslant \epsilon(2\mu+\lambda)4k|\Omega|/(k-2)^{\gamma+1}$. However, in order to obtain the last inequality, we also need $\gamma > 3$ and k to be so large that $k^{1-\gamma/3} \leqslant C(1-2/k)^{\gamma+1}$. (We denote the last condition by (**).)

Considering, as usual, a variational formulation of Eq. (9) with test functions from the space $\mathbf{D}_\sigma^{1,2}(\Omega) \oplus \{\nabla\psi;\, \psi \in W^{2,2}(\Omega)\}$, and passing to the limit as $\epsilon \to 0$ for fixed large k, we prove the next theorem:

Theorem 1. *Let $m > 0$ and $\gamma > 3$. Let $\mathbf{f} \in \mathbf{L}^r(\Omega)$ (where $3 < r < +\infty$) be given. Then there exists a weak solution $(\mathbf{v}, \varphi, \rho, \Pi)$ to the problem* (NS) *in $\mathbf{D}_\sigma^{1,q}(\Omega) \times W^{2,q}(\Omega) \times L^\infty(\Omega) \times L^q(\Omega)$ for all $1 \leqslant q < +\infty$. Moreover, ρ satisfies the estimate $\rho \leqslant k-1$ a.e. in Ω, provided that k is so large that it fulfills both the conditions $k > m/|\Omega|$ and (**).*

Let us denote, for a while, by ρ_n the density, corresponding to $\epsilon_n > 0$, chosen so that $\epsilon_n \searrow 0$ for $n \to +\infty$. The classical difficulty in the proof concerns the limit transition (for $n \to +\infty$) in the term $\mathcal{P}_k(\rho_n)$, which is essentially bounded in Ω. One must show that the weak-star limit equals $\mathcal{P}(\rho)$, where ρ is the weak-star limit of an appropriate subsequence of $\{\rho_n\}$.

5. Regularity of the weak solution

Let $(\mathbf{v}, \varphi, \rho, \Pi)$ be a weak solution to the problem (NS), given by Theorem 1. Denoting $\mathbf{F} := \rho\mathbf{f} - \rho((\mathbf{v}+\nabla\varphi)\cdot\nabla)(\mathbf{v}+\nabla\varphi)$, we observe that the part $\mathbf{v}$ of the solution satisfies

$$\mu \int_\Omega \mathbf{curl}\,\mathbf{v} \cdot \mathbf{curl}\,\boldsymbol{\phi}\, d\mathbf{x} = \int_\Omega \mathbf{F} \cdot \boldsymbol{\phi}\, d\mathbf{x} \tag{19}$$

for all $\boldsymbol{\phi} \in \mathbf{D}_\sigma^{1,2}(\Omega)$. Using the inclusion $\mathbf{F} \in \mathbf{L}^r(\Omega)$ and treating (19) as a variational formulation of the Stokes equation $\mu\,\mathbf{curl}^2\,\mathbf{v} + \nabla\Pi = \mathbf{F}$, we can improve the information on the regularity of the functions $\mathbf{v}$ and Π. Thus, we obtain:

Theorem 2. *Let $(\mathbf{v}, \varphi, \rho, \Pi)$ be a weak solution to the problem* (NS)*, given by Theorem* 1*. Then $\mathbf{v} \in \mathbf{D}_\sigma^{2,r}(\Omega)$ and $\Pi \in W^{1,r}(\Omega)$.*

6. Discussion

The Navier–Stokes equations for incompressible fluid with the boundary conditions (4) have already been treated and solved e.g. in [1] and [8]. In the compressible case, the weak solvability of the system (1), (2) has been studied by many authors (e.g. by P.L. Lions and A. Novotný), mostly with the no-slip boundary conditions. Articles [6,7] and this paper show that the system (1), (2) with the boundary conditions (4) represents an alternative model.

The case $\gamma > 5/3$ was studied in [6], where the existence of a weak solution $(\rho, \mathbf{u})$ in $L^{s(\gamma)}(\Omega) \times \mathbf{W}^{1,2}(\Omega)$ (where $s(\gamma) = 2\gamma$ if $\gamma \geqslant 3$ and $s(\gamma) = 3(\gamma-1)$ if $5/3 < \gamma < 3$). The result can be extended to $\gamma > 3/2$ if $\mathbf{curl}\,\mathbf{f} = \mathbf{0}$; see [5].

A presently solved model for $\gamma > 3$ provides a weak solution with the part $\mathbf{v}$, Π having the regularity typical for strong solutions. This is obtained by techniques, which is closely connected with the boundary conditions (4). Moreover, the higher regularity of $\mathbf{v}$ enables us to justify the boundary condition (4(iii)) as an intrinsic property of the solution.

The introduced regularized model is more concise than other usually applied approximate models. For example, we do not need artificial corrections of the pressure as e.g. in [9]. The chosen regularization turns out to be a good way to obtain a solution with a bounded density. This interesting fact was already observed by several authors, e.g. P. Mucha and M. Pokorný in the case of Navier's slip boundary conditions, see [4].

Finally, we mention the paper [2] by J. Frehse et al., where the existence of a weak solution to the Dirichlet problem is established for an improved lower bound for γ, i.e. $\gamma > 4/3$. It is realistic to expect that, following the treatment of the "b-norms" $\|\rho^b|\mathbf{u}|^2\|_1$ in the same way as in [2], our results can also be extended to lower values of γ in the pressure term ρ^γ.

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