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Mathematical Analysis

Functions of perturbed operators

Aleksei Aleksandrov [a], Vladimir Peller [b]

[a] *St-Petersburg Branch, Steklov Institute of Mathematics, Fontanka 27, 191023 St-Petersburg, Russia*
[b] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*

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Abstract

We prove that if $0<\alpha<1$ and f is in the Hölder class $\Lambda_\alpha(\mathbb{R})$, then for arbitrary selfadjoint operators A and B with bounded $A-B$, the operator $f(A)-f(B)$ is bounded and $\|f(A)-f(B)\|\leqslant \text{const}\,\|A-B\|^\alpha$. We prove a similar result for functions f of the Zygmund class $\Lambda_1(\mathbb{R})$: $\|f(A+K)-2f(A)+f(A-K)\|\leqslant \text{const}\,\|K\|$, where A and K are selfadjoint operators. Similar results also hold for all Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{R})$, $\alpha>0$. We also study properties of the operators $f(A)-f(B)$ for $f\in\Lambda_\alpha(\mathbb{R})$ and selfadjoint operators A and B such that $A-B$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$. We consider the same problem for higher order differences. Similar results also hold for unitary operators and for contractions. ***To cite this article: A. Aleksandrov, V. Peller, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Fonctions d'opérateurs perturbés. Nous démontrons que si $0<\alpha<1$ et f appartient à la classe de Hölder $\Lambda_\alpha(\mathbb{R})$, alors pour tous les opérateurs A et B autoadjoints dont la différence est bornée on a : $\|f(A)-f(B)\|\leqslant \text{const}\,\|A-B\|^\alpha$. Nous obtenons un résultat analogue pour les fonctions de la classe de Zygmund $\Lambda_1(\mathbb{R})$: $\|f(A+K)-2f(A)+f(A-K)\|\leqslant \text{const}\,\|K\|$, où A et K sont des opérateurs autoadjoints. Un résultat analogue est aussi vrai pour toutes les classes de Hölder–Zygmund $\Lambda_\alpha(\mathbb{R})$, $\alpha>0$. Nous étudions aussi les propriétés des opérateurs $f(A)-f(B)$ si $f\in\Lambda_\alpha(\mathbb{R})$ et A et B sont des opérateurs autoadjoints dont la différence appartient à la classe de Schatten–von Neumann $\boldsymbol{S}_p$. Nous considérons le même problème pour les différences d'ordre arbitraire. On peut obtenir des résultats analogues pour les opérateurs unitaires et pour les contractions. ***Pour citer cet article : A. Aleksandrov, V. Peller, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Il est bien connu qu'il existe des fonctions f lipschitziennes sur la droite réelle $\mathbb{R}$ qui ne sont pas lipschitziennes opératorielles, c'est-à-dire la condition,

$$\big|f(x)-f(y)\big|\leqslant \text{const}\,|x-y|,\quad x,y\in\mathbb{R},$$

E-mail address: peller@math.msu.edu (V. Peller).

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n'implique pas que pour tous les opérateurs autoadjoints A et B soit vraie l'inégalité :

$$\|f(A) - f(B)\| \leqslant \text{const}\,\|A - B\|.$$

Il se trouve que la situation change dramatiquement si l'on considère les fonctions de la classe $\Lambda_\alpha(\mathbb{R})$ de Hölder d'ordre α, $0 < \alpha < 1$. Nous démontrons que si A et B sont des opérateurs autoadjoints dans un espace hilbertien et $f \in \Lambda_\alpha(\mathbb{R})$ (c'est-à-dire $|f(x) - f(y)| \leqslant \text{const}\,|x - y|^\alpha$), alors

$$\|f(A) - f(B)\| \leqslant \text{const}\,\|A - B\|^\alpha.$$

On peut considérer un problème analogue pour la classe de Zygmund $\Lambda_1(\mathbb{R})$ des fonctions continues sur $\mathbb{R}$ telles que

$$\big|f(x+t) - 2f(x) + f(x-t)\big| \leqslant \text{const}\,|t|, \quad x, t \in \mathbb{R}.$$

Nous établissons que dans ce cas f doit satisfaire à l'inégalité,

$$\big\|f(A+K) - 2f(A) + f(A-K)\big\| \leqslant \text{const}\,\|K\|,$$

où A et K sont des opérateurs autoadjoints.

Nous considérons aussi les espaces $\Lambda_\alpha(\mathbb{R})$ pour tous les $\alpha > 0$. Si $\alpha > 0$, la classe $\Lambda_\alpha(\mathbb{R})$ est celle des fonctions f continues telles que

$$\left|\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kt)\right| \leqslant \text{const}\,|t|^\alpha \quad (\text{ici } n - 1 \leqslant \alpha < n).$$

Dans ce cas nous montrons :

$$\left\|\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(A+jK)\right\| \leqslant \text{const}\,\|K\|^\alpha, \quad n > \alpha,$$

pour tout couple d'opérateurs autoadjoints A et K.

De plus, on peut généraliser les résultats ci-dessus pour les opérateurs A non nécessairement bornés (Théorème 4.2 dans la version anglaise).

On peut considérer les mêmes problèmes pour les opérateurs unitaires et pour les contractions.

Nous considérons aussi le cas de modules de continuité arbitraires. Soit ω une fonction croissante continue sur $[0, \infty)$ telle que $\omega(0) = 0$ et $\omega(x+y) \leqslant \omega(x) + \omega(y)$, $x, y \geqslant 0$. Nous définissons la fonction ω^* par :

$$\omega^*(x) = x \int_x^\infty \frac{\omega(t)}{t^2}\,\mathrm{d}t, \quad x \geqslant 0.$$

Notons par $\Lambda_\omega(\mathbb{R})$ l'espace de fonctions f sur $\mathbb{R}$ telles que

$$\big|f(x) - f(y)\big| \leqslant \text{const}\,\omega\big(|x - y|\big), \quad x, y \in \mathbb{R}.$$

Nous démontrons que si A et B sont des opérateurs autoadjoints dont la différence $A - B$ est bornée et $f \in \Lambda_\omega$, alors

$$\big\|f(A) - f(B)\big\| \leqslant \text{const}\,\omega^*\big(\|A - B\|\big).$$

On peut obtenir un résultat similaire pour les modules de continuité d'ordre arbitraire. Nous avons aussi obtenu les mêmes résultats pour les opérateurs unitaires et pour les contractions.

Maintenant nous considérons le problème suivant. Rappelons que $\boldsymbol{S}_p$ est la classe de Schatten–von Neumann constituée des opérateurs T dans un espace hilbertien pour lesquels les nombres singuliers $s_n(T)$ appartiennent à l'espace ℓ^p.

Supposons que $f \in \Lambda_\alpha(\mathbb{R})$, $0 < \alpha < 1$, et $p > 1$. Nous montrons que pour tout couple d'opérateurs autoadjoints A et B dont la différence $A - B$ appartient à $\boldsymbol{S}_p$, on a :

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{p}{\alpha}} \quad \text{et} \quad \big\|f(A) - f(B)\big\|_{\boldsymbol{S}_{\frac{p}{\alpha}}} \leqslant \text{const}\,\|A - B\|^\alpha.$$

Si $p = 1$ et $f \in \Lambda_\alpha(\mathbb{R})$, $0 < \alpha < 1$, la condition $A - B \in \boldsymbol{S}_1$ implique que

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{1}{\alpha},\infty},$$

où l'espace $\boldsymbol{S}_{q,\infty}$ est formé des opérateurs dont les nombres singuliers satisfont à la condition :

$$\sup_{n\geqslant 0}(1+n)^{1/q}s_n(T) < \infty.$$

Nous obtenons des résultats analogues pour les différences d'ordre arbitraire. On peut obtenir des résultats semblables pour les opérateurs unitaires et pour les contractions.

1. Introduction

It is well known that a Lipschitz function on the real line is not necessarily *operator Lipschitz*, i.e., the condition,

$$\left|f(x) - f(y)\right| \leqslant \text{const}\,|x-y|, \quad x, y \in \mathbb{R},$$

does not imply that for selfadjoint operators A and B on Hilbert space,

$$\left\|f(A) - f(B)\right\| \leqslant \text{const}\,\|A-B\|.$$

The existence of such functions was proved in [4] (see also [5] and [7]). Later in [8] necessary conditions were found for a function f to be operator Lipschitz. Those necessary conditions imply that Lipschitz functions do not have to be operator Lipschitz. It is also well known that a continuously differentiable function does not have to be operator differentiable, see [8] and [9]. Note that the necessary conditions obtained in [8] and [9] are based on the nuclearity criterion for Hankel operators, see [10].

It turns out that the situation dramatically changes if we consider Hölder classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily *operator Hölder of order* α, i.e., the condition:

$$\left|f(x) - f(y)\right| \leqslant \text{const}\,|x-y|^\alpha, \quad x, y \in \mathbb{R},$$

implies that for selfadjoint operators A and B on Hilbert space,

$$\left\|f(A) - f(B)\right\| \leqslant \text{const}\,\|A-B\|^\alpha. \tag{1}$$

Moreover, a similar result holds for the Zygmund class $\Lambda_1(\mathbb{R})$, i.e., the fact that

$$\left|f(x+t) - 2f(x) + f(x-t)\right| \leqslant \text{const}\,|t|, \quad x, t \in \mathbb{R},$$

and f is continuous implies that f is *operator Zygmund*, i.e., for selfadjoint operators A and K,

$$\left\|f(A+K) - 2f(A) + f(A-K)\right\| \leqslant \text{const}\,\|K\|. \tag{2}$$

We also obtain similar results for the whole scale of Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{R})$ for $0 < \alpha < \infty$. Recall that for $\alpha > 1$, the class $\Lambda_\alpha(\mathbb{R})$ consists of continuous functions f such that

$$\left|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+kt)\right| \leqslant \text{const}\,|t|^\alpha, \quad \text{where } n-1 \leqslant \alpha < n.$$

The same problems can be considered for unitary operators and for functions on the unit circle, and for contractions and analytic functions in the unit disk.

To prove (1), we use a crucial estimate obtained for trigonometric polynomials and unitary operators in [8] and for entire functions of exponential type and selfadjoint operators in [9]. We state here the result for selfadjoint operators. It can be considered as an analog of Bernstein's inequality.

Let f be an entire function of exponential type σ that is bounded on the real line $\mathbb{R}$. Then for selfadjoint operators A and B with bounded $A - B$ the following inequality holds:

$$\left\|f(A) - f(B)\right\| \leqslant \text{const}\,\sigma\,\|f\|_{L^\infty(\mathbb{R})}\|A-B\|. \tag{3}$$

Inequality (3) was proved by using double operator integrals and the Birman–Solomyak formula:

$$f(A)-f(B)=\iint \frac{f(x)-f(y)}{x-y}\,\mathrm{d}E_A(x)(A-B)\,\mathrm{d}E_B(y),$$

where E_A and E_B are the spectral measures of selfadjoint operators A and B; we refer the reader to [1], [2] and [3] for the theory of double operator integrals. Note that A and B do not have to be bounded, but $A-B$ must be bounded.

To estimate the second difference (2), we use the corresponding analog of Bernstein's inequality which was obtained in [11] with the help of triple operator integrals. To estimate higher order differences, we need multiple operator integrals. We refer the reader to [11] for definitions and basic results on multiple operator integrals.

We also consider in this paper the problem of the behavior of functions of operators $f(A)$ under perturbations of A by operators of Schatten–von Neumann class $\boldsymbol{S}_p$ in the case when $f\in\Lambda_\alpha(\mathbb{R})$.

2. Norm estimates for unitary operators

We start with first order differences. We use the notation by Λ_α, $0<\alpha<\infty$, for the scale of Hölder–Zygmund classes on the unit circle $\mathbb{T}$.

Theorem 2.1. *Let $0<\alpha<1$. Then there is a constant $c>0$ such that for every $f\in\Lambda_\alpha$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds*:

$$\big\|f(U)-f(V)\big\|\leqslant c\|f\|_{\Lambda_\alpha}\cdot\|U-V\|^\alpha.$$

Theorem 2.2. *There exists a constant $c>0$ such that for every function $f\in\Lambda_1$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds*:

$$\big\|f(U)-f(V)\big\|\leqslant c\|f\|_{\Lambda_1}\left(2+\log_2\frac{1}{\|U-V\|}\right)\|U-V\|.$$

Note that this result improves an estimate obtained in [4] for Lipschitz functions in the case of bounded selfadjoint operators.

We proceed now to higher order differences.

Theorem 2.3. *Let n be a positive integer and $0<\alpha<n$. Then there exists a constant $c>0$ such that for every $f\in\Lambda_\alpha$ and for an arbitrary unitary operator U and an arbitrary bounded selfadjoint operator A on Hilbert space the following inequality holds*:

$$\left\|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f\big(e^{\mathrm{i}kA}U\big)\right\|\leqslant c\|f\|_{\Lambda_\alpha}\|A\|^\alpha.$$

Let us consider now a more general problem. Suppose that ω is a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0)=0$ and $\omega(x+y)\leqslant\omega(x)+\omega(y)$, $x,y\geqslant 0$. The space Λ_ω consists of functions f on $\mathbb{T}$ such that

$$\big|f(\zeta)-f(\tau)\big|\leqslant \operatorname{const}\omega\big(|\zeta-\tau|\big),\quad \zeta,\tau\in\mathbb{T}.$$

With a modulus of continuity ω we associate the function ω^* defined by:

$$\omega^*(x)=x\int_x^\infty\frac{\omega(t)}{t^2}\,\mathrm{d}t,\quad x\geqslant 0.$$

Theorem 2.4. *Suppose that ω is a modulus of continuity and $f\in\Lambda_\omega$. If U and V are unitary operators, then*

$$\big\|f(U)-f(V)\big\|\leqslant\operatorname{const}\|f\|_{\Lambda_\omega}\,\omega^*\big(\|U-V\|\big).$$

In particular, if $\omega^*(x) \leqslant \text{const}\,\omega(x)$, then for unitary operators U and V

$$\big\| f(U) - f(V) \big\| \leqslant \text{const}\, \|f\|_{\Lambda_\omega} \omega\big(\|U - V\|\big).$$

We have also proved an analog of Theorem 2.4 for higher order differences.

3. Norm estimates for contractions

We denote here by $(\Lambda_\alpha)_+$ the set of functions $f \in \Lambda_\alpha$, for which the Fourier coefficients $\hat{f}(n)$ vanish for $n < 0$.

Recall that an operator T on Hilbert space is called a contraction if $\|T\| \leqslant 1$. The following result is an analog of Theorem 2.3 for contractions.

Theorem 3.1. *Let n be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds*:

$$\left\| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(T + \frac{k}{n}(T - R)\right) \right\| \leqslant c \|f\|_{\Lambda_\alpha} \|T - R\|^\alpha.$$

Note that an analog of Theorem 2.4 also holds for contractions.

4. Norm estimates for selfadjoint operators

Theorem 4.1. *Let $0 < \alpha < 1$ and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B$ is bounded. Then $f(A) - f(B)$ is bounded and*

$$\big\| f(A) - f(B) \big\| \leqslant \text{const}\, \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\|^\alpha.$$

In this connection we mention the paper [4] where it was proved that for selfadjoint operators A and B with spectra in an interval $[a, b]$ and a function $\varphi \in \Lambda_\alpha(\mathbb{R})$, the following inequality holds:

$$\big\|\varphi(A) - \varphi(B)\big\| \leqslant \text{const}\, \|\varphi\|_{\Lambda_\alpha(\mathbb{R})} \left(\log\left(\frac{b-a}{\|A - B\|} + 1 \right) + 1 \right)^2 \|A - B\|^\alpha$$

(see also [6] where the above inequality is generalized for general moduli of continuity).

Theorem 4.2. *Suppose that n is a positive integer and $0 < \alpha < n$. Let A be a selfadjoint operator and let K be a bounded selfadjoint operator. Then the map,*

$$f \mapsto \big(\Delta_K^n f\big)(A) \stackrel{\text{def}}{=} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(A + jK), \tag{4}$$

has a unique extension from $L^\infty \cap \Lambda_\alpha(\mathbb{R})$ to a sequentially continuous operator from $\Lambda_\alpha(\mathbb{R})$ (equipped with the weak-star topology) to the space of bounded linear operators on Hilbert space (equipped with the strong operator topology) and

$$\big\| \big(\Delta_K^n f\big)(A) \big\| \leqslant \text{const}\, \|f\|_{\Lambda_\alpha(\mathbb{R})} \|K\|^\alpha.$$

We use the same notation $(\Delta_K^n f)(A)$ for the unique extension of the map (4).

We can also prove an analog of Theorem 2.4 for selfadjoint operators.

5. Perturbations of class $\boldsymbol{S_p}$

In this section we consider the behavior of functions of selfadjoint operators under perturbations of Schatten–von Neumann class $\boldsymbol{S}_p$. Similar results also hold for unitary operators and for contractions.

Recall that the spaces $\boldsymbol{S}_p$ and $\boldsymbol{S}_{p,\infty}$ consist of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S}_p} \stackrel{\text{def}}{=} \left(\sum_{n\geqslant 0}\big(s_n(T)\big)^p\right)^{1/p} < \infty \quad \text{and} \quad \|T\|_{\boldsymbol{S}_{p,\infty}} \stackrel{\text{def}}{=} \sup_{n\geqslant 0}(1+n)^{1/p}s_n(T) < \infty.$$

Theorem 5.1. *Let $1 \leqslant p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B \in \boldsymbol{S}_p$. Then*

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{p}{\alpha},\infty} \quad \textit{and} \quad \big\|f(A) - f(B)\big\|_{\boldsymbol{S}_{\frac{p}{\alpha},\infty}} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^\alpha_{\boldsymbol{S}_p}.$$

Note that in Theorem 5.1 in the case $p > 1$ we can replace the condition $A - B \in \boldsymbol{S}_p$ with the condition $A - B \in \boldsymbol{S}_{p,\infty}$.

Using interpolation arguments, we can deduce from Theorem 5.1 the following result:

Theorem 5.2. *Let $1 < p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B \in \boldsymbol{S}_p$. Then*

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{p}{\alpha}} \quad \textit{and} \quad \big\|f(A) - f(B)\big\|_{\boldsymbol{S}_{\frac{p}{\alpha}}} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^\alpha_{\boldsymbol{S}_p}.$$

Let us now state similar results for higher order differences.

Theorem 5.3. *Suppose that n is a positive integer, α is a positive number such that $n - 1 \leqslant \alpha < n$, and $n \leqslant p < \infty$. Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S}_p$. Then the operator $(\Delta^n_K f)(A)$ defined in Theorem* 4.2 *belongs to $\boldsymbol{S}_{\frac{p}{\alpha},\infty}$, and*

$$\big\|\big(\Delta^n_K f\big)(A)\big\|_{\boldsymbol{S}_{\frac{p}{\alpha},\infty}} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha_{\boldsymbol{S}_p}.$$

Theorem 5.4. *Suppose that n is a positive integer, α is a positive number such that $n - 1 \leqslant \alpha < n$, $f \in \Lambda_\alpha(\mathbb{R})$, and $n < p < \infty$. Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S}_p$. Then the operator $(\Delta^n_K f)(A)$ defined in Theorem* 4.2 *belongs to $\boldsymbol{S}_{\frac{p}{\alpha}}$, and*

$$\big\|\big(\Delta^n_K f\big)(A)\big\|_{\boldsymbol{S}_{\frac{p}{\alpha}}} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha_{\boldsymbol{S}_p}.$$

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