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Group Theory

On Tits' Centre Conjecture for fixed point subcomplexes

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Abstract

We give a short and uniform proof of a special case of Tits' Centre Conjecture using a theorem of J.-P. Serre and a result from the authors in 2005. We consider fixed point subcomplexes X^H of the building $X = X(G)$ of a connected reductive algebraic group G, where H is a subgroup of G. ***To cite this article: M. Bate et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur la conjecture du centre de Tits pour les sous-complexes de points fixes. Nous donnons dans cette Note une démonstration courte et uniforme d'un cas particulier de la conjecture du centre de Tits, en utilisant un théorème de J.-P. Serre et un résultat des auteurs en 2005. Nous considérons les sous-complexes X^H de l'immeuble $X = X(G)$ associé à un groupe connexe réductif G, des points fixes de l'action d'un sous-groupe H de G. ***Pour citer cet article : M. Bate et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k. Let $X = X(G)$ be the spherical Tits building of G, cf. [10]. Recall that the simplices in X correspond to the parabolic subgroups of G, [8, §3.1]; for a parabolic subgroup P of G, we let x_P denote the corresponding simplex of X. The conjugation action of G on itself naturally induces an action of G on the building X, so the image of G is a subgroup of the automorphism group of X. Given a subcomplex Y of X, let $N_G(Y)$ denote the subgroup of G consisting of elements which stabilize Y (in this induced action).

Recall the *geometric realization* of X as a bouquet of n-spheres. A subcomplex Y of X is called *convex* if whenever two points of Y (in the geometric realization) are not opposite in X, then Y contains the unique geodesic joining these points, [8, §2.1]. A convex subcomplex Y of X is *contractible* if it has the homotopy type of a point, [8, §2.2]. The following is a version due to J.-P. Serre of the so-called "Centre Conjecture" by J. Tits, cf. [9, Lem. 1.2], [6, §4],

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[8, §2.4], [11]. This has been proved by B. Mühlherr and J. Tits for spherical buildings of classical type [5]. The simplex referred to in the conjecture is called a *centre* for Y.

Conjecture 1.1. *Let Y be a convex contractible subcomplex of X. Then there is a simplex in Y which is stabilized by all automorphisms of X which stabilize Y.*

For a subgroup H of G let X^H be the fixed point subcomplex of the action of H, i.e., X^H consists of the simplices $x_P \in X$ such that $H \subseteq P$. Thus, if $H \subseteq K \subseteq G$ are subgroups of G, then we have $X^K \subseteq X^H$; observe that X^H is always convex, cf. [8, Prop. 3.1]. Our main result, Theorem 3.1, gives a short, conceptual proof of a special case of Conjecture 1.1; namely, we consider subcomplexes of the form $Y = X^H$ for H a subgroup of G, and we consider automorphisms only from $N_G(Y)$. The special case $G = \mathrm{GL}(V)$ in Theorem 3.1 generalizes the classical construction of upper and lower Loewy series, see Remark 3.2(ii).

The initial motivation for Tits' Conjecture 1.1 was a question about the existence of a certain parabolic subgroup associated with a unipotent subgroup of a Borel subgroup of G (cf. [6, §4.1], [8, §2.4]). This existence theorem was ultimately proved by other means, [3, §3]. In Example 3.6 below we show that the existence of such a parabolic subgroup can be viewed as a special case of Theorem 3.1.

2. Serre's notion of complete reducibility

Following Serre [8, Def. 2.2.1], we say that a convex subcomplex Y of X is *X-completely reducible* (X-cr) if for every simplex $y \in Y$ there exists a simplex $y' \in Y$ opposite to y in X. The following is part of a theorem due to Serre, [6, Thm. 2]; see also [8, §2] and [11]:

Theorem 2.1. *Let Y be a convex subcomplex of X. Then Y is X-completely reducible if and only if Y is not contractible.*

The notion of convexity for subcomplexes of X has the following nice characterization in terms of parabolic subgroups due to Serre, [8, Prop. 3.1]:

Proposition 2.2. *Let Y be a subcomplex of X. Then Y is convex if and only if whenever P, P', and Q are parabolic subgroups in G with $x_P, x_{P'} \in Y$ and $Q \supseteq P \cap P'$, then $x_Q \in Y$.*

Note that many subcomplexes which arise naturally in the building are fixed point subcomplexes. For example, the apartments of X are the subcomplexes X^T for maximal tori T of G and, more generally, the smallest convex subcomplex containing two simplices x_P and $x_{P'}$ is $X^{P \cap P'}$.

Following Serre [8], we say that a (closed) subgroup H of G is *G-completely reducible* (G-cr) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P; for an overview of this concept see for instance [7] and [8]. In the case $G = \mathrm{GL}(V)$ (V a finite-dimensional k-vector space) a subgroup H is G-cr exactly when V is a semisimple H-module, so this faithfully generalizes the notion of complete reducibility from representation theory. An important class of G-cr subgroups consists of those that are not contained in any proper parabolic subgroup of G at all (they are trivially G-cr). Following Serre, we call them *G-irreducible* (G-ir), [8]. As before, in the case $G = \mathrm{GL}(V)$, this concept coincides with the usual notion of irreducibility. If H is a G-completely reducible subgroup of G, then H^0 is reductive, [7, Property 4].

Since X^H is a convex subcomplex of $X = X(G)$ for any subgroup H of G, Theorem 2.1 applies in this case and we have the following result (see [7, p. 19], [8, §3]):

Theorem 2.3. *Let H be a subgroup of G. Then H is G-completely reducible if and only if the subcomplex X^H is not contractible.*

Remark 2.4. By convention, the empty subcomplex of X is not contractible. This is consistent with Theorem 2.1, because H is G-ir if and only if $X^H = \emptyset$, and a G-ir subgroup is G-cr.

Our next result [1, Thm. 3.10] gives an affirmative answer to a question by Serre, [7, p. 24]. The special case when $G = \mathrm{GL}(V)$ is just a particular instance of Clifford Theory.

Theorem 2.5. *Let $N \subseteq H \subseteq G$ be subgroups of G with N normal in H. If H is G-completely reducible, then so is N.*

3. Tits' Centre Conjecture for fixed point subcomplexes

Here is the main result of this Note:

Theorem 3.1. *Let Y be a convex, contractible subcomplex of X. Suppose that Y is of the form $Y = X^H$ for a subgroup H of G. Then there is a simplex in Y which is stabilized by all elements in $N_G(Y)$.*

Proof. Let M be the intersection of all parabolic subgroups of G corresponding to simplices in Y. Since $H \subseteq M$, we have $X^M \subseteq X^H$. But every parabolic subgroup containing H contains M, by definition of M. Hence $X^M = X^H$. Set $K := N_G(Y)$. It is clear that M is normal in K. Since $X^K \subseteq X^M$, it suffices to show that $X^K \neq \emptyset$. Now $Y = X^M$ is contractible, so Theorem 2.3 implies that M is not G-cr. Thus, by Theorem 2.5, it follows that K is not G-cr and again by Theorem 2.3 that X^K is contractible. In particular, X^K is non-empty, by Remark 2.4. Thus K stabilizes a simplex in X^M, as claimed. □

Remarks 3.2. (i). Let $H \subseteq K \subseteq G$ be subgroups of G with H normal in K. Suppose that X^H is contractible. Since H is normal in K, the latter permutes the simplices in X^H, and so $K \subseteq N_G(X^H)$. It thus follows from Theorem 3.1 that K fixes a simplex in X^H.

(ii). Observe that Theorem 3.1 can be viewed as a generalization of the classical construction of upper and lower Loewy series in representation theory (for definitions, see e.g., [4]). Let V be a finite-dimensional k-vector space. Let $H \subseteq K \subseteq \mathrm{GL}(V)$ be subgroups of $\mathrm{GL}(V)$ with H normal in K and suppose that V is not H-semisimple. Then the upper and lower Loewy series of the H-module V are proper K-stable flags in V, and so they provide "natural centres" for the action of K on the complex $X(V)^H$, where $X(V)$ is the flag complex of V.

(iii). In [8, Prop. 2.11], J.-P. Serre showed that Theorem 2.5 is a consequence of Tits' Centre Conjecture 1.1. So, Theorem 3.1 is just the reverse implication of Serre's result [8, Prop. 2.11] in the special case when Theorem 2.5 applies.

(iv). Let k_0 be any field and let k be the algebraic closure of k_0. Suppose that G is defined over k_0. One can define what it means for a subgroup H defined over k_0 to be G-completely reducible over k_0, cf. [1, Sec. 5], [8, Sec. 3]. In [1, Thm. 5.8], it is proved that if k_0 is perfect, then a subgroup H is G-cr over k_0 if and only if it is G-cr. Using this, one can show that the proof of Theorem 3.1 goes through for buildings of the form $X = X(G(k_0))$. In particular, this includes many finite spherical buildings attached to finite groups of Lie type.

(v). In the Centre Conjecture 1.1, one considers all automorphisms of the building. If $X = X(G)$, then in many cases, $\mathrm{Aut}\, X$ is generated by inner and graph automorphisms of G, together with field automorphisms (cf. [10, Intro.]). We will consider graph and field automorphisms in the setting of Theorem 3.1 in future work (see [2, Sec. 5]).

Our final result gives a characterization of subcomplexes of X of the form X^H for a subgroup H of G.

Proposition 3.3. *Let $Y \subseteq X$ be a subset of simplices of X. Then Y is a subcomplex of X of the form $Y = X^H$ for some subgroup H of G if and only if for every $n \in \mathbb{N}$, the following condition holds*:

(3.4) *if $P_1, \ldots, P_n$, Q are parabolic subgroups with $x_{P_i} \in Y$ and $Q \supseteq \bigcap_{i=1}^n P_i$, then $x_Q \in Y$.*

Proof. First suppose that $Y = X^H$ for some subgroup H of G. Let $n \in \mathbb{N}$ and let $x_{P_1}, \ldots, x_{P_n} \in Y$. If Q is a parabolic subgroup of G containing $\bigcap_{i=1}^n P_i$, then Q contains H, because each P_i does, so $x_Q \in Y$.

Conversely, suppose that condition (3.4) holds for all $n \in \mathbb{N}$. Let H be the intersection of all P such that $x_P \in Y$. By the descending chain condition, we have $H = \bigcap_{i=1}^m P_i$ for some $m \in \mathbb{N}$ and some P_i with $x_{P_i} \in Y$. It follows from condition (3.4) for $n = m$ that for any parabolic subgroup P containing H, $x_P \in Y$, so $X^H \subseteq Y$. It is clear from the definition of H that $Y \subseteq X^H$. □

Remark 3.5. Note that Y is a subcomplex of X precisely when condition (3.4) holds for $n = 1$. Further, by Proposition 2.2, Y is convex if and only if condition (3.4) holds for $n = 2$.

As indicated in the introduction, a fundamental theorem of Borel and Tits on unipotent subgroups of Borel subgroups of G [3, §3] yields a key example for Theorem 3.1.

Example 3.6. Let U be a non-trivial unipotent subgroup of G contained in a Borel subgroup B of G. Let $Y = X^U$. Note that U is not G-cr; for if U is contained in a Borel subgroup B^- opposite to B, then U is contained in the maximal torus $B^- \cap B$ of G, which is absurd. So Y is contractible, by Theorem 2.3. Thus, by Theorem 3.1, $N_G(U)$ stabilizes a simplex in Y, i.e., there is a parabolic subgroup P of G containing $N_G(U)$. Now, the construction of Borel and Tits in [3] yields such a parabolic subgroup P which enjoys additional properties; for example, it is stabilized by automorphisms of G which stabilize U. The framework for G-complete reducibility developed in [1] and subsequent papers allows one to associate such *canonical* parabolic subgroups to all non-G-cr subgroups of G, see [2, Sec. 5].

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