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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 211–216
COMPTES RENDUS MATHEMATIQUE

Mathematical Problems in Mechanics/Calculus of Variations

A nonlinear theory for shells with slowly varying thickness

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Received 16 April 2008; accepted 16 December 2008

Available online 10 February 2009

Presented by Philippe G. Ciarlet

Abstract

We study the Γ-limit of 3d nonlinear elasticity for shells of small, variable thickness, around an arbitrary smooth 2d surface. ***To cite this article: M. Lewicka et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une théorie des coques d'épaisseurs faiblement variables. On étudie la Γ-limite de la théorie de l'élasticité non linéaire pour une coque mince à épaisseur variable autour d'une surface arbitraire de dimension 2. ***Pour citer cet article : M. Lewicka et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cet article, on étudie la Γ-limite de l'énergie élastique non linéaire pour une coque mince à épaisseur variable, de surface moyenne de dimension 2 dans $\mathbb{R}^3$.

Le problème tridimensionnel : Soit S une surface compacte, connexe et orientable dans $\mathbb{R}^3$ de regularité $C^{1,1}$. On suppose que le bord de S est une union finie de courbes lipschitziennes. Dans ce qui suit, $\vec{n}(x)$, T_xS, et $\Pi(x) = \nabla\vec{n}(x)$ sont respectivement la normale unitaire, l'espace tangent, et la deuxième forme fondamentale de S evalués au point $x \in S$. Soient $g_1, g_2 : S \to \mathbb{R}$ deux fonctions positives, lipschitziennes, bornées par un paramètre donné suffisamment petit. On considère une famille de coques minces :

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -hg_1(x) < t < hg_2(x)\} \quad \text{pour } 0 < h < 1.$$

L'énergie élastique d'une déformation $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ est définie par : $E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$, où la densité de l'énergie $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ est une fonction $\mathcal{C}^2$ dans un voisinage de SO(3), qui satisfait (1).

Le problème bidimensionnel : On introduit l'espace $\mathcal{V}$ des *isométries infinitésimales* $V \in W^{2,2}(S, \mathbb{R}^3)$, c.à.d. les champs de vecteurs V pour lesquels il existe une application $A \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ tels que $\partial_\tau V(x) = A(x)\tau$ et $A(x)^T =$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.12.017

$-A(x)$ pour tout $\tau \in T_x S$, p.p. dans S. Soit *l'espace des déformations finies* $\mathcal{B}$ (voir [7]) défini comme adhérence dans la topologie de L^2 de l'espace de champs de matrices symétriques sur S sous forme de $(\mathrm{sym}\nabla w)$ pour un déplacement quelconque $w \in W^{1,2}(S, \mathbb{R}^3)$.

L'énergie de von Kármán $I(V, B_{\tan})$ est donnée pour $V \in \mathcal{V}$ et $B_{\tan} \in \mathcal{B}$ par l'expression :

$$I(V, B_{\tan}) = \frac{1}{2}\int_S (g_1 + g_2)\mathcal{Q}_2\Big(x, B_{\tan} - \frac{\kappa}{2}\big(A^2\big)_{\tan} - \frac{1}{2}\mathrm{sym}\big(A\nabla\big((g_2 - g_1)\vec{n}\big)\big)\Big)$$
$$+ \frac{1}{24}\int_S (g_1 + g_2)^3 \mathcal{Q}_2\big(x, \big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\big).$$

La forme quadratique $\mathcal{Q}_2(x,\cdot)$ est définie par $\mathcal{Q}_2(x, F_{\tan}) = \min\{\mathcal{Q}_3(\tilde{F}); (\tilde{F} - F)_{\tan} = 0\}$ où $\mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F, F)$. Elle dépend seulement de la partie symétrique de $F_{\tan}$, et restreinte à la classe des matrices symétriques elle est définie positive. Soit e^h une suite de nombres positifs telle que : $\lim_{h\to 0} e^h/h^4 = \kappa^2 < \infty$. Le résultat fondamental identifie la Γ-limite des fonctionnelles $(1/e^h)E^h$:

Théorème 0.1.

(a) *Soit* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *une suite de déformations vérifiant* (4). *Alors il existe des suites* $Q^h \in \mathrm{SO}(3)$ *et* $c^h \in \mathbb{R}^3$, *pour lesquelles les* $y^h(x + t\vec{n}) = (Q^h)^T u^h(x + ht\vec{n}) - c^h$, *définies sur le domaine commun* S^1, *aient les propriétés suivantes* :
  (i) y^h *converge dans* $W^{1,2}(S^1)$ *vers la projection* $\pi : S^1 \to S$ *le long la normale* $\vec{n}$;
  (ii) *Une sous-suite des déplacements en moyenne* (5) *converge dans* $W^{1,2}(S)$ *vers* $V \in \mathcal{V}$;
  (iii) *Une sous-suite de* $\frac{1}{h}\mathrm{sym}\nabla V^h[y^h]$ *converge faiblement dans* $L^2(S)$ *vers* $B_{\tan} \in \mathcal{B}$;
  (iv) $\liminf_{h\to 0} \frac{1}{e^h} E^h(u^h) \geqslant I(V, B_{\tan})$.

(b) *Réciproquement, pour tout* $V \in \mathcal{V}$ *et* $B_{\tan} \in \mathcal{B}$, *il existe une suite* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$, *vérifiant* (4) *et* (i), (ii), (iii) (*avec* $Q^h = \mathrm{Id}$ *et* $c^h = 0$), *et de plus* : $\lim_{h\to 0} \frac{1}{e^h} I^h(y^h) = I(V, B_{\tan})$.

Cette analyse peut être aussi traitée en présence de conditions au bord ou de forces extérieurs. Dans cette Note, on considère une suite de forces $f^h \in L^2(S, \mathbb{R}^3)$ en S, avec les propriétés données par (8). On définit leurs extensions sur S^h par $f^h(x + t\vec{n}) = \det(\mathrm{Id} + t\Pi(x))^{-1} f^h(x)$. Soit m^h l'action maximisée de f^h pour toutes les rotations de S^h : $m^h = \max_{Q\in\mathrm{SO}(3)} \frac{1}{h}\int_{S^h} f^h(z)\cdot Qz\,\mathrm{d}z$. Alors, la fonctionnelle de l'énergie totale J^h sur S^h est donnée par $J^h(u^h) = E^h(u^h) + m^h - \frac{1}{h}\int_{S^h} f^h u^h$. Par conséquent : $0 \geqslant \inf\{\frac{1}{e^h} J^h(y^h) ; u^h \in W^{1,2}(S^h, \mathbb{R}^3)\} \geqslant -C$. On introduit la relaxation $r : \mathrm{SO}(3) \to [0, \infty]$ par (9).

Théorème 0.2. *Supposons encore* (3) *et* (8). *Soit* $u^h \in W^{1,2}(S, \mathbb{R}^3)$ *une suite minimisante de* $\frac{1}{e^h} J^h$, *c.à.d. qu'elle vérifie* (10). *Alors les conclusions de Theoreme* 0.1(a) *sont vraies, et tout point d'accumulation* $\bar{Q}$ *de* $\{Q^h\}$ *appartient à* $\mathcal{M} = \{\bar{Q} \in \mathrm{SO}(3) ; r(\bar{Q}) < \infty\}$. *De plus, toute limite* $(V, B_{\tan}, \bar{Q})$ *minimise la fonctionelle :* $J(V, B_{\tan}, \bar{Q}) = I(V, B_{\tan}) - \int_S (g_1 + g_2) f \cdot \bar{Q}V + r(\bar{Q})$, *définie pour* $(V, B_{\tan}, \bar{Q}) \in \mathcal{V} \times \mathcal{B} \times \mathcal{M}$.

1. Introduction

The following question receives large attention in the current literature on elasticity [1]: which theories of thin objects (rods, plates, shells) are predicted by 3d nonlinear theory? For plates, this problem has been extensively studied through Γ-convergence; first by Le Dret and Raoult [5], leading to a rigorous derivation of the membrane theory, and later by Friesecke, James and Müller [4], for the Kirchhoff, von Kármán and linear theories. In this framework, much less has been done for shells. The membrane and the bending theories were obtained in [6] and [2], respectively. More recently, the generalized von Kármán and linear theories have been rigorously introduced and justified by the authors in [7]. In this Note, we present these last new results, in an extended version for shells with variable thickness.

2. The setting of the problem and the related elastic energy functionals

2.1. The three-dimensional problem

Let S be a 2d compact, connected, oriented surface in $\mathbb{R}^3$, of class $\mathcal{C}^{1,1}$, whose boundary is the union of finitely many (possibly none) Lipschitz curves. Let $\vec{n}(x)$ be the unit normal vector, T_xS the tangent space, and $\Pi(x) = \nabla\vec{n}(x)$ the shape operator on S, at a given $x \in S$.

For given positive Lipschitz functions $g_1, g_2 : S \to (0, \infty)$, small in L^∞, consider a family of thin shells:

$$S^h = \bigl\{z = x + t\vec{n}(x);\ x \in S,\ -hg_1(x) < t < hg_2(x)\bigr\}, \quad 0 < h < 1.$$

The *elastic energy* (scaled per unit thickness) of a deformation $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ is given by $E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$, where the stored-energy density $W : \mathbb{R}^{3\times3} \to [0, \infty]$ is $\mathcal{C}^2$ in a neighborhood of SO(3), and satisfies:

$$\forall F \in \mathbb{R}^{3\times3}\ \forall R \in \mathrm{SO}(3) \quad W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geqslant C\,\mathrm{dist}^2\bigl(F, \mathrm{SO}(3)\bigr). \tag{1}$$

2.2. The two-dimensional problem

We now introduce the *von Kármán functional* on S:

$$\begin{aligned} I(V, B_{\tan}) = {} & \frac{1}{2}\int_S (g_1 + g_2)\mathcal{Q}_2\Bigl(x, B_{\tan} - \frac{\kappa}{2}\bigl(A^2\bigr)_{\tan} - \frac{1}{2}\mathrm{sym}\bigl(A\nabla\bigl((g_2 - g_1)\vec{n}\bigr)\bigr)\Bigr) \\ & + \frac{1}{24}\int_S (g_1 + g_2)^3\mathcal{Q}_2\bigl(x, \bigl(\nabla(A\vec{n}) - A\Pi\bigr)_{\tan}\bigr), \end{aligned} \tag{2}$$

defined for $V \in \mathcal{V}$ and $B_{\tan} \in \mathcal{B}$. The space $\mathcal{V}$ consists of *infinitesimal isometries* $V \in W^{2,2}(S, \mathbb{R}^3)$, that is the vector fields V such that there exists a matrix field $A \in W^{1,2}(S, \mathbb{R}^{3\times3})$ satisfying:

$$\partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x)^T = -A(x) \quad \forall \text{a.e. } x \in S\ \forall\tau \in T_xS.$$

The *finite strain space* $\mathcal{B}$ (see [7]), consists of the following symmetric matrix fields:

$$\mathcal{B} = \Bigl\{L^2 - \lim_{h\to0} \mathrm{sym}\,\nabla w^h;\ w^h \in W^{1,2}\bigl(S, \mathbb{R}^3\bigr)\Bigr\}.$$

In (2) $\kappa \geqslant 0$ is a parameter, and the positive definite quadratic forms $\mathcal{Q}_2(x, \cdot)$ are defined as follows:

$$\mathcal{Q}_2(x, F_{\tan}) = \min\bigl\{\mathcal{Q}_3(\tilde{F});\ (\tilde{F} - F)_{\tan} = 0\bigr\}, \qquad \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F, F).$$

The form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times3}$, while $\mathcal{Q}_2(x, \cdot)$, for $x \in S$, is defined on tangential minors $F_{\tan}$ of such matrices. Recall that the tangent space to SO(3) at Id is so(3). As a consequence [3], both forms depend only on the symmetric parts of their arguments, and are positive definite on symmetric matrices.

2.3. Relative stretching and bending

The two terms in (2) are strictly tied to the deformations of the *geometric mid-surface* of S^h. Given $V \in \mathcal{V}$ and a field $w \in W^{1,2}(S, \mathbb{R}^3)$, consider the deformations:

$$\tilde{\phi}^h = \mathrm{id} + 1/2h(g_2 - g_1)\vec{n}, \qquad \phi^h = \tilde{\phi}^h + hV + h^2w.$$

Then $\tilde{\phi}^h(S)$ is the geometric mid-surface. A straightforward calculation shows that for any $\tau \in T_xS$:

$$\bigl|\partial_\tau\phi^h\bigr|^2 - \bigl|\partial_\tau\tilde{\phi}^h\bigr|^2 = 2h^2\tau^T\Bigl(\mathrm{sym}\,\nabla w - \frac{1}{2}A^2 - \frac{1}{2}\mathrm{sym}\bigl(A\nabla\bigl((g_2 - g_1)\vec{n}\bigr)\bigr)\Bigr)\tau + \mathcal{O}\bigl(h^3\bigr).$$

Hence (putting $\kappa = 1$ and $B_{\tan} = \mathrm{sym}\,\nabla w$), the expression in the argument of $\mathcal{Q}_2$ in the first term of (2) describes *stretching*, namely the second order in h change of the first fundamental form of $\tilde{\phi}^h(S)$.

The second term of (2) relates to *bending*, that is the first order in h change in the second fundamental form of $\tilde{\phi}^h(S)$. To see this, let Π^h be the shape operator on $\phi^h(S)$. Indeed, for $\tau \in T_xS$ one obtains:

$$\bigl(\nabla\phi^h\bigr)^{-1}\Pi^h\bigl(\nabla\phi^h\bigr)\tau - \bigl(\nabla\tilde{\phi}^h\bigr)^{-1}\Pi\bigl(\nabla\tilde{\phi}^h\bigr)\tau = h\bigl(\partial_\tau(A\vec{n}) - A\Pi\tau\bigr) + \mathcal{O}\bigl(h^2\bigr).$$

Notice also, that the term $-\frac{1}{2}\,\mathrm{sym}(A\nabla((g_2-g_1)\vec{n}))$ which is new with respect to analysis in [7], disappears when $g_1=g_2$ or equivalently $S=\tilde{\phi}^h(S)$. This term expresses the first order in h deficit for V from being an infinitesimal isometry on $\tilde{\phi}^h(S)$. It is so because for any $\eta=\partial_\tau\tilde{\phi}^h(x)$ where $\tau\in T_xS$ we have:

$$\partial_\eta\big(V\circ\big(\tilde{\phi}^h\big)^{-1}\big)\cdot\eta=\partial_\tau V\cdot\partial_\tau\tilde{\phi}^h=-h/2\tau^T\,\mathrm{sym}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)\tau.$$

3. The Γ-convergence of the elastic energy functionals

Theorem 3.1. *Let e^h be a sequence of positive numbers, for which we assume*:

$$\lim_{h\to 0}\frac{e^h}{h^4}=\kappa^2<\infty. \tag{3}$$

(a) *For any sequence of deformations $u^h\in W^{1,2}(S^h,\mathbb{R}^3)$ satisfying*:

$$E^h\big(u^h\big)\leqslant Ce^h, \tag{4}$$

there exist a sequence $Q^h\in\mathrm{SO}(3)$ and $c^h\in\mathbb{R}^3$ such that for the normalized rescaled deformations: $y^h(x+t\vec{n})=(Q^h)^Tu^h(x+ht\vec{n})-c^h$ *defined on the common domain S^1, the following holds*:

(i) *y^h converge in $W^{1,2}(S^1)$ to the projection $\pi:S^1\to S$ along the normal $\vec{n}$*;

(ii) *The related scaled average displacements,*

$$\big(V^h\big[y^h\big]\big)(x)=\frac{h}{\sqrt{e^h}}\fint_{-g_1(x)}^{g_2(x)}y^h(x+t\vec{n})-(x+ht\vec{n})\,\mathrm{d}t, \tag{5}$$

converge (*up to a subsequence*) *in $W^{1,2}(S)$ to some $V\in\mathcal{V}$*;

(iii) $\frac{1}{h}\,\mathrm{sym}\,\nabla V^h[y^h]$ *converge* (*up to a subsequence*) *weakly in $L^2(S)$ to some $B_{\tan}\in\mathcal{B}$*;

(iv) $\liminf_{h\to 0}\frac{1}{e^h}E^h(u^h)\geqslant I(V,B_{\tan})$.

(b) *Conversely, for every $V\in\mathcal{V}$ and $B_{\tan}\in\mathcal{B}$, there exists a sequence $u^h\in W^{1,2}(S^h,\mathbb{R}^3)$ satisfying* (4) *and such that* (i), (ii), (iii) *hold* (*with $Q^h=\mathrm{Id}$ and $c^h=0$*) *and moreover*: $\lim_{h\to 0}\frac{1}{e^h}E^h(y^h)=I(V,B_{\tan})$.

Proof. Here we outline the main steps of the proof of Theorem 3.1. We refer the readers to [7] for a detailed exposition in the case $g_1=g_2=\mathrm{const}$. All convergences below are up to a subsequence.

1. A first major ingredient is approximating u^h by $W^{1,2}$ matrix fields $R^h:S\to\mathrm{SO}(3)$, thanks to the scaling invariant, nonlinear quantitative rigidity estimate from [3]. Noting (4) and the uniform bound of the functional $\int\mathrm{dist}^2(\nabla\cdot,\mathrm{SO}(3))$ by $E^h(\cdot)$ on the uniformly Lipschitz domains S^h, it follows that:

$$\big\|\nabla u^h-R^h\pi\big\|_{L^2(S^h)}\leqslant Ch^{1/2}\sqrt{e^h},\qquad \big\|\nabla R^h\big\|_{L^2(S)}+\big\|R^h-Q^h\big\|_{L^2(S)}\leqslant Ch^{-1}\sqrt{e^h},$$

for some $Q^h\in\mathrm{SO}(3)$. Further, for some skew-symmetric matrix field $A\in W^{1,2}(S,\mathbb{R}^3)$, one has:

$$h/\sqrt{e^h}\big(\big(Q^h\big)^TR^h-\mathrm{Id}\big)\to A\quad\text{weakly in }W^{1,2}(S),$$
$$h^2/e^h\,\mathrm{sym}\big(\big(Q^h\big)^TR^T-\mathrm{Id}\big)\to 1/2A^2\quad\text{strongly in }L^2(S).$$

2. Define now $w^h(x+t\vec{n})=h/\sqrt{e^h}((Q^h)^Tu^h(x+ht\vec{n})-c^h-(x+ht\vec{n}))$, with c^h such that $\int_{S^1}w^h=0$. To prove (i) and (ii), one first uses the above convergences to obtain:

$$\nabla_{\tan}w^h\to A\pi(\mathrm{Id}+t\Pi)^{-1}\quad\text{and}\quad\partial_{\vec{n}}w^h\to 0,\quad\text{strongly in }L^2\big(S^1\big).$$

By the Poincaré inequality, w^h must now converge to $V\circ\pi$, strongly in $W^{1,2}(S^1)$.

For (iii), one identifies various terms in $\frac{1}{h}\mathrm{sym}\nabla V^h[y^h]$ to notice that it converges weakly in $L^2(S)$ to:

$$\fint_{-g_1}^{g_2}(\mathrm{sym}\,G)_{\tan}\,\mathrm{d}t+\frac{\kappa}{2}\big(A^2\big)_{\tan}+\frac{1}{2}\,\mathrm{sym}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)=:B_{\tan}, \tag{6}$$

where G is the weak $L^2(S^1)$ limit of the matrix fields $G^h(x+t\vec{n}) = \frac{1}{\sqrt{e^h}}((R^h)^T\nabla u^h(x+ht\vec{n}) - \mathrm{Id})$. The convergence of G^h [7], which is the $\sqrt{e^h}$ order term in the expansion of the nonlinear strain $(\nabla u^h)^T\nabla u^h$ at Id, plays a major role for the expansion of the energy density: $W(\nabla u^h) = W(\mathrm{Id} + \sqrt{e^h}\,G^h)$.

3. Towards proving (iv), the previously established convergences and the definition of $\mathcal{Q}_2$ imply:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{e^h}E^h(u^h) &\geqslant \frac{1}{2}\int_S\int_{-g_1}^{g_2}\mathcal{Q}_2\big(x,(\operatorname{sym}G)_{\tan}\big)\,\mathrm{d}t\,\mathrm{d}x\\ &= \frac{1}{2}\int_S\int_{-(g_1+g_2)/2}^{(g_1+g_2)/2}\mathcal{Q}_2\big(x,(\operatorname{sym}G_0)_{\tan} + \big(s+(g_2-g_1)/2\big)(\nabla_{\tan}A)\vec{n}\big)\,\mathrm{d}s\,\mathrm{d}x,\end{aligned}\tag{7}$$

where we used the fact that $(\operatorname{sym}G)_{\tan}(x+t\vec{n}) = (\operatorname{sym}G_0)_{\tan}(x) + t(\nabla_{\tan}A)\vec{n}$ is linear in t. Calculating $(\operatorname{sym}G_0)_{\tan}$ from (6) and collecting various terms in the argument of the quadratic form $\mathcal{Q}_2$ above, we see that the right hand side in (7) equals the von Kármán functional $I(V, B_{\tan})$.

4. If $V \in \mathcal{V}\cap W^{2,\infty}(S,\mathbb{R}^3)$ and $B_{\tan} = \operatorname{sym}\nabla w$ with $w \in W^{2,\infty}(S,\mathbb{R}^3)$, then the recovery sequence is given by the formula below. When V and $B_{\tan}$ do not have the required regularity, one proceeds by approximation and truncation, as in [7].

$$\begin{aligned}y^h(x+t\vec{n}) = x &+ \frac{h}{2}(g_2-g_1)\vec{n}(x) + \sqrt{e^h}/h\,V(x) + \sqrt{e^h}\,w(x) + h\Big(t-\frac{1}{2}(g_2-g_1)\Big)\vec{n}(x)\\ &+ \Big(t-\frac{1}{2}(g_2-g_1)\Big)\sqrt{e^h}\big(\Pi V_{\tan} - \nabla(V\vec{n})\big)(x) - h\Big(t-\frac{1}{2}(g_2-g_1)\Big)\sqrt{e^h}\,\vec{n}^T\nabla w\\ &+ \Big(t-\frac{1}{2}(g_2-g_1)\Big)h\sqrt{e^h}\,d^0(x) + \frac{1}{2}\Big(t-\frac{1}{2}(g_2-g_1)\Big)^2 h\sqrt{e^h}\,d^1(x).\end{aligned}$$

The vector fields $d^0, d^1 \in W^{1,\infty}(S,\mathbb{R}^3)$ are given by means of the linear map $c(x, F_{\tan})$ which returns the unique vector c satisfying $\mathcal{Q}_2(x, F_{\tan}) = \min\{\mathcal{Q}_3(F_{\tan} + c\otimes\vec{n}(x) + \vec{n}(x)\otimes c);\ \ c\in\mathbb{R}^3\}$. Then

$$\begin{aligned}d^0 &= 2c\Big(x, B_{\tan} - \frac{\kappa}{2}(A^2)_{\tan} - \frac{1}{2}\operatorname{sym}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)\Big) + \kappa A^2\vec{n} - \frac{\kappa}{2}\big(\vec{n}^TA^2\vec{n}\big)\vec{n} + \frac{1}{2}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)^T\vec{n},\\ d^1 &= 2c\big(x, \operatorname{sym}\big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\big) + \vec{n}^TA\Pi - \vec{n}^T\nabla(A\vec{n}). \qquad \square\end{aligned}$$

4. The dead loads and convergence of minimizers

Consider a sequence of forces $f^h \in L^2(S,\mathbb{R}^3)$ acting on S, with the following properties:

$$\int_S (g_1+g_2)f^h = 0 \quad\text{and}\quad \lim_{h\to 0}\frac{1}{h\sqrt{e^h}}f^h = f \quad\text{weakly in } L^2(S).\tag{8}$$

Define their extensions $f^h(x+t\vec{n}) = \det(\mathrm{Id}+t\Pi(x))^{-1}f^h(x)$ over S^h, and the maximized action of f^h over all rotations of S^h as $m^h = \max_{Q\in SO(3)}\frac{1}{h}\int_{S^h}f^h(z)\cdot Qz\,\mathrm{d}z$. The total energy functional on S^h is:

$$J^h(u^h) = E^h(u^h) + m^h - \frac{1}{h}\int_{S^h}f^hu^h.$$

As in [7], one can prove that $0 \geqslant \inf\{\frac{1}{e^h}J^h(y^h);\ u^h \in W^{1,2}(S^h,\mathbb{R}^3)\} \geqslant -C$. We further introduce the relaxation function $r : SO(3) \to [0,\infty]$, with its effective domain $\mathcal{M} = \{\bar{Q}\in SO(3);\ r(\bar{Q}) < \infty\}$:

$$r(Q) = \inf\bigg\{\liminf_{h\to 0}\frac{1}{e^h}\bigg(m^h - \frac{1}{h}\int_{S^h}f^h\cdot Q^hz\,\mathrm{d}z\bigg);\ Q^h\in SO(3),\ Q^h\to Q\bigg\}.\tag{9}$$

Theorem 4.1. *Assume* (3) *and* (8). *Let* $u^h \in W^{1,2}(S, \mathbb{R}^3)$ *be any minimizing sequence of* $\frac{1}{e^h} J^h$, *that is*:

$$\lim_{h\to 0}\left(\frac{1}{e^h} J^h\left(y^h\right) - \inf \frac{1}{e^h} J^h\right) = 0. \tag{10}$$

Then the conclusions of Theorem 3.1(a) *hold, and any accumulation point* $\bar{Q}$ *of* $\{Q^h\}$ *belongs to* $\mathcal{M}$. *Moreover, any limit* $(V, B_{\tan}, \bar{Q})$ *minimizes the following functional, over* $V \in \mathcal{V}$, $B_{\tan} \in \mathcal{B}$ *and* $\bar{Q} \in \mathcal{M}$:

$$J(V, B_{\tan}, \bar{Q}) = I(V, B_{\tan}) - \int_S (g_1 + g_2) f \cdot \bar{Q} V + r(\bar{Q}).$$

The proof follows as in [7]. An equivalent formulation of Theorem 4.1 in terms of Γ-convergence is possible.

Example. When $f^h = h\sqrt{e^h} f$ and $g_1 = g_2$, then $\mathcal{M} = \{\bar{Q} \in \mathrm{SO}(3);\ \int_S f \cdot \bar{Q}x = \max_{Q\in \mathrm{SO}(3)} \int_S f \cdot Qx\}$ (in the general case the inclusion $\subset$ is still true, but not the other one). Further, $r \equiv 0$ on $\mathcal{M}$, so the term $r(\bar{Q})$ may be dropped in the definition of J. In the general case, both r and $\mathcal{M}$ depend on the asymptotic behavior of the maximizers of the linear functions $\mathrm{SO}(3) \ni Q \mapsto \int_{S^h} f^h(z) \cdot Qz\, \mathrm{d}z$.

4.1. Approximately robust surfaces and higher scalings

Some classes of surfaces, including surfaces of revolution, developable surfaces with no flat part, and elliptic surfaces, have the property to recompense the second order stretching (of $\tilde{\phi}^h(S)$) introduced by the infinitesimal isometry V, through a suitable second order displacement. As an example, surfaces with flat parts do not enjoy this property (for any g_1, g_2). For such surfaces [7], described by condition: $\{A^2 + \mathrm{sym}(A\nabla((g_2 - g_1)\vec{n}));\ V \in \mathcal{V}\} \subset \mathcal{B}$, the limit elastic energy $I(V, B_{\tan})$ simplifies and should be replaced (in Theorems 3.1 and 4.1, regardless of $\kappa \geqslant 0$) by:

$$\tilde{I}(V) = \frac{1}{24} \int_S (g_1 + g_2)^3 \mathcal{Q}_2\big(x, \big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\big). \tag{11}$$

When $\kappa = 0$ and $g_1 = g_2$, the stretching term is negligible for any S and hence the von Kármán theory again reduces to the linear theory, with the elastic energy given in (11). For more discussion, see [7].

Acknowledgements

M.L. was partially supported by the NSF grant DMS-0707275. M.G.M. was partially supported by GNAMPA through the project "Problemi di riduzione di dimensione per strutture elastiche sottili" 2008.

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