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Nonsingular Ricci flow on a noncompact manifold in dimension three

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Abstract

We consider the Ricci flow $\frac{\partial}{\partial t} g = -2Ric$ on the 3-dimensional complete noncompact manifold $(M, g(0))$ with nonnegative curvature operator, i.e., $Rm \geqslant 0$, and $|Rm(p)| \to 0$, as $d(o, p) \to \infty$. We prove that the Ricci flow on such a manifold is nonsingular in any finite time. ***To cite this article: L. Ma, A. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Flot de Ricci non singulier sur une variété tridimensionnelle non compacte. Nous considérons le flot de Ricci $\frac{\partial}{\partial t} g = -2Ric$ sur la variété tridimensionnelle complète de courbure non négatif, c'est-à-dire $Rm \geqslant 0$ et $|Rm(p)| \to 0$ si $d(o, p) \to \infty$. Nous démontrons que le flot de Ricci sur une telle variété est non singular pour tout temps fini. ***Pour citer cet article : L. Ma, A. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note nous considérons le flot de la courbure de Ricci $\frac{\partial g}{\partial t}(t) = -2\,\mathrm{Ricci}_{g(t)}$ sur des variétés complètes de dimension 3 et d'opérateur de courbure positif ou nul. Nous supposons que le tenseur de Riemann, $|\mathrm{Riem}(p)|$, tend vers 0 lorsque le point p tend vers l'infini. Nous démontrons alors que le flot de Ricci est défini pour tout temps $t > 0$ et est non singulier. Ce type de questions a été posé par R. Hamilton. Sachant que l'existence en temps petit des solutions est prouvée par W.-X. Shi, il ne reste qu'à montrer que la courbure est bornée sur tout intervalle de temps fini. Plus précisément dans ce travail nous prouvons le théorème suivant :

Théorème 0.1. *Supposons que $(M, g(t))$ est un flot de Ricci pour $t \in [0, T)$ sur une variété de dimension 3 complète non compacte, connexe. Nous supposons que l'opérateur de courbure de $(M, g(0))$ est positif ou nul et vérifie $|\mathrm{Riem}(p, g(0))| \to 0$ lorsque $d(0, p) \to +\infty$. Alors $T = +\infty$, c'est-à-dire que le flot est non singulier sur tout intervalle de temps fini.*

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Par le théorème de l'âme de Cheeger–Gromoll et Meyer la variété est difféomorphe à $\mathbf{R}^3$. Rappelons deux remarques importantes :

1) Si $\mathrm{Riem}(x,t) := \mathrm{Riem}_{g(t)}(x)$ a une valeur propre nulle, alors, par le principe du maximum fort, la métrique le long du flot relevé au revêtement universel se décompose en un produit et la condition sur la courbure ne peut être satisfaite que si la métrique est constamment plate. Dans ce cas, il est évident que le flot existe pour tout temps.
2) La convergence lorsque $t \to +\infty$ est illustrée par les exemples donnés en introduction et est plus subtile que pour le cas ou t tend vers une limite finie.

Nous utilisons de manière importante les notions et idées introduites par G. Perelman dans ses travaux et nous nous appuyons sur les détails fournis dans l'ouvrage récent de J. Morgan et G. Tian.

1. Introduction

The aim of this Note is to get a global existence of Ricci flow with bounded nonnegative curvature operator in three dimensions. This kind of question was asked by Hamilton [6]. We remark that the local existence of the flow was obtained by Shi [15]. So we only need to show that the curvature is bounded in finite time. Our research is based on previous important results obtained by Hamilton and Perelman [6,9,10], which will be recalled in next section.

The Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ on a compact manifold was first introduced by Richard Hamilton [7]. Using it, Hamilton had obtained an remarkable theorem [7] that a compact 3-manifold with positive Ricci curvature can be deformed by the Ricci flow to a space form. Then we met a useful program, the so called Hamilton's program, which is to prove Poincaré conjecture and Thurston's geometrization conjecture by Ricci flow. In three remarkable papers [12–14], Perelman significantly advanced the theory of the Ricci flow. Perelman introduced important results such as a noncollapsing, canonical neighborhood, and analysis of the high curvature regions. Perelman also analyzed one of the special solution to the Ricci flow, the κ solution, which is usually the limit solution of the blow up sequence. Before the works of Perelman, Hamilton [6] had defined asymptotic volume for a complete noncompact manifold, and he had obtained that the asymptotic volume is constant under Ricci flow with bounded curvature. By an induction argument, Perelman obtained that the asymptotic volume is zero when the solution is an κ solution. In order to analyze the high curvature region, Hamilton obtained a very interesting compactness result of Ricci flow [8]. However, in order to apply this compactness, one has to check the assumptions of noncollapsing and bounded curvature. We shall use ideas above to study nonsingular Ricci flow on a complete noncompact Riemannian manifold of dimension three. The purpose of this work is to show that the following result is true:

Theorem 1.1. *Assume that $(M, g(t)), t \in [0, T)$ is a Ricci flow on the 3-dimensional connected complete noncompact Riemannian manifold $(M, g(0))$. Suppose the curvature operator of the initial metric $g(0)$ is positive, i.e., $Rm(g(0)) \geqslant 0$ with $|Rm(p, g(0))| \to 0$, $d(o, p) \to \infty$. Then $T = \infty$, i.e., Ricci flow is nonsingular in finite time on such a manifold.*

By the Soul theorem (Cheeger–Gromoll–Meyer, see Theorem 2.7 in p. 56 in [10]), each $(M, g(t))$ is diffeomorphic to $\mathbf{R}^3$.

We make two remarks here:

(1) If $Rm(x,t) := Rm(x, g(t))$ has a zero eigenvalue, then, by the strong maximum principle, we can split the flow on the level of its covering space. Then the condition $|Rm(p)| \to 0$, as $d(o, p) \to \infty$, cannot be satisfied, unless the manifold is flat; in this case the Ricci flow exists for all time (see Corollary 4.20 in [10]).
(2) We point out that in our proof of Theorem 1.1, the convergence means the geometric convergence (Definition 5.12 in p. 114 in [10]). However, as for the convergence of the global flow as $t \to \infty$, we have following interesting example, which shows that the convergence question is subtle:

Example 1.2. Consider the revolution paraboloid $x_4 = x_1^2 + x_2^2 + x_3^2$, where $(x_1, \ldots, x_4) \in R^4$. We know that its curvature operator satisfies $Rm(x) \to 0$, $x \to \infty$ and $Rm > 0$. By Theorem 1.1, the Ricci flow on it can a nonsingular global flow. Using Hamilton's result mentioned above, it is not hard to see that the asymptotic volume of the flow is 0, so the paraboloid cannot converge to flat $\mathbf{R}^3$ as $t \to \infty$ in any topology which preserves the asymptotic volume.

Example 1.3. Let us consider another example often used by physicists. Let the 3-dimensional Ricci flow $(M, g(t))$ satisfy the assumptions of Theorem 1.1. Assume also that it is also asymptotical to a cone $(\mathbf{R}^3 - B_R(0))/\Gamma$ (where $R \gg 1$ and Γ is a finite subgroup of O(3)) at infinity for each fixed time t, which implies that the asymptotic volume is between 0 and ω, where ω is the volume of unit ball $B(0,1) \subset \mathbf{R}^3$. Again, we have a global flow, but we cannot have any convergence result of the flow as $t \to \infty$ in any topology which preserves the asymptotic volume.

We remark that in the radial symmetrical case, a similar result was obtained in [11], where another assumption such as the asymptotic flatness was used.

Remark 1.4. X. Dai and L. Ma proved that the Ricci flow on the asymptotically flat manifold cannot converge uniformly to the flat manifold by using ADM mass (see [4]). One may see our previous work [4] for more results in this direction.

2. Preliminary results

In this section, we recall the deep results of Perelman, Hamilton, and others, which will be needed in the proof of Theorem 1.1.

By the monotonicity of $\mathcal{W}$ functional and the reduced volume (see [12] for the definitions), Perelman proved the noncollapsing of Ricci flow on a compact manifold. Furthermore, Perelman obtained the following convergence theorems about Ricci flow on 3-manifolds (see [12,9], and in particular Chapter 11 in [10] for more detail), which we will use:

Theorem 2.1. *Fix canonical neighborhood constants* (C, ϵ), ([10, *pp.* 239–241]) *and noncollapsing constants* $r > 0$, $\kappa > 0$. *Let* $(\mathcal{M}_n, G_n, x_n)$ *be a sequence of based* 3*-dimensional Ricci flows* (*the same result is true for based generalized* 3*-dimensional Ricci flow, for its definition, see* [10], *Definition* 3.37 *in p.* 87). *We set* $t_n = t(x_n)$ *and* $Q_n = R(x_n)$. *We denote by* M_n *the time* t_n *time slice of* $\mathcal{M}_n$. *We assume*:

- (i) *Each* $(\mathcal{M}_n, G_n)$ *has time interval of definition contained in* $[0, \infty)$ *and has curvature pinched toward positive* ([10], *p.* 251, *Definition* 10.1);
- (ii) *Every point* $y_n \in (\mathcal{M}_n, G_n)$ *with* $t(y_n) \leqslant t_n$ *and* $R(y_n) \geqslant 4R(x_n)$ *has a strong* (C, ϵ) *canonical neighborhood*;
- (iii) $\lim_{n\to\infty} Q_n = \infty$;
- (iv) *For each* $A < \infty$, *for all n sufficiently large, the ball* $B(x_n, t_n, AQ_n^{-1/2})$ *has compact closure in* M_n *and the flow is* κ *noncollapsed on scales* $\leqslant r$ *at each point of* $B(x_n, t_n, AQ_n^{-1/2})$;
- (v) *There is* $\mu > 0$ *such that for every* $A < \infty$ *the following holds for all n sufficiently large, if* $y_n \in B(x_n, t_n, AQ^{-1/2})$ *the maximum flow line through* y_n *extends backwards for a time at least* $\mu(\max(Q_n, R(y_n)))^{-1}$.

Then for a subsequence and shifting the times of each Ricci flow so that $t_n = 0$ *for every n, there is a geometric limit* $(M_\infty, g_\infty, x_\infty)$ *of the sequence of based Riemannian manifolds* $(M_n, Q_nG_n(0), x_n)$. *The limit is a complete* 3*-dimensional Riemannian manifold of bounded, nonnegative curvature. Furthermore, for some* $t_0 > 0$ *depending on the curvature bound for* (M_∞, g_∞), *there is a geometric limiting Ricci flow* (*see Definition* 5.12 *in* [10]) *defined on* $(M_\infty, g_\infty(t))$, $-t_0 \leqslant t \leqslant 0$, *with* $g_\infty(0) = g_\infty$.

Remark 2.2. We observe that in this theorem, we do not need that the time interval contains a fixed subinterval in order to avoid shrinking to a point. However, we need condition (v) to substitute it.

We now borrow the following result from Theorem 11.8 in [10]:

Theorem 2.3. *Suppose that* $\{\mathcal{M}_n, G_n, x_n\}_{n=1}^{\infty}$ *is a sequence of* 3*-dimensional Ricci flows satisfying all the hypothesis of Theorem* 2.1. *Let us assume in addition that there is* T_0 *with* $0 < T_0 \leqslant \infty$ *such that the following holds. For any*

$T < T_0$, for each $A < \infty$, and all n sufficiently large, there is an embedding $B(x_n, t_n, AQ_n^{-1/2}) \times (t_n - TQ_n^{-1}, t_n]$ into $\mathcal{M}_n$ compatible with time and with the vector field and for every point of the image the flow is κ noncollapsed on scales $\leqslant r$. Then, after shifting the times of the generalized flows such that $t_n = 0$ for all n and for a subsequence there is a geometric limiting Ricci flow

$$\big(M_\infty, g_\infty(t), x_\infty\big), \quad -T_0 < t \leqslant 0,$$

for the rescaled flows $(\mathcal{M}_n, Q_n G_n, x_n)$. This limiting flow is complete with a nonnegative curvature. Furthermore, the curvature is locally bounded in time. If in addition $T_0 = \infty$, then it is a κ solution.

Remark 2.4. For the application of Theorem 2.3 above to our case, we need to explain a little more. In fact, we do not need the generalized Ricci flow since we have not considered Ricci flow with surgery. Hence, the sentence "there is an embedding $B(x_n, t_n, AQ_n^{-1/2}) \times (t_n - TQ_n^{-1}, t_n]$ into $\mathcal{M}_n$ compatible with time and with the vector field" in Theorem 2.3 means that in our case, a Ricci flow defined on $B(x_n, t_n, AQ_n^{-1/2}) \times (t_n - TQ_n^{-1}, t_n]$. We remark that the sentence "curvature locally bounded in time" means that for any $T < T_0$, the curvature is uniformly bounded on the time interval $(-T, 0]$ by a positive constant $C(T)$, which depends only on T.

We also need the following result on the Ricci flow (see Theorem 1 in [1]):

Theorem 2.5. *Let (M^n, g) be a complete noncompact Riemannian manifold with injectivity radius bounded away from zero such that $|Rm|(x) \to 0$ as $x \to \infty$. Let $(M, g(t))$ be the corresponding maximal solution to the Ricci flow on $M \times [0, T)$. Then either $T = \infty$ or there exists some compact $S \subset M$ such that $|Rm(x,t)|$ is bounded on $(M - S) \times [0, T)$.*

3. Proof of Theorem 1.1

In the following, we consider the Ricci flow on a complete noncompact Riemannian manifold $(M, g(0))$ of dimension three with its curvature operator $Rm > 0$, $|Rm(p)| \to 0$, as $d(o, p) \to \infty$, where o is a fixed point. In order to applied Theorem 2.3 above, we need two results below. The method used to prove them comes from Perelman's famous papers (see [12,13]). However, the condition there is a little different from ours. For the definition of κ-noncollapsed on the scale $\rho > 0$, we refer to Definition 4.2 in [12].

Lemma 3.1. *For sufficiently small $r > 0$, there is $\kappa > 0$ such that the Ricci flow $(M, g(t))$, $t \in [0, T)$, on our complete noncompact manifold $(M, g(0))$ is κ noncollapsed on the scale $\rho \leqslant r$.*

Proof. Since $Rm(g(0) > 0$, Shi [15] has proved that the positivity of curvature operator is preserved by the Ricci flow $(M, g(t))$, $t \in (0, T)$. By a well known result of Gromoll and Meyer ([5], [2] see also Theorem B65 in p. 312 in [3] with its proof), we have an injectivity radius estimate $\mathrm{inj}(M^n, g(t)) \geqslant \pi/\sqrt{R_{\max}(t)}$.

Fix a point $(x, t_0) \in M \times [0, T)$. Since we have the bound for the scalar curvature $R(y, t) < C_1$, $(y, t) \in M \times [0, \frac{1}{2}t_0]$, by the above injectivity radius estimate, we have $\mathrm{Vol}\, B(y, t, r) \geqslant V' r^3$, $(y, t) \in M \times [0, \frac{1}{2}t_0]$. where V' is a positive constant, which may depends on t.

The following computation can be found in [12] (see Chapter 6 and Chapter 9 in [10] and also the interesting paper of Ye [16]).

By the inequality of reduced length

$$\frac{\partial l_x}{\partial \tau}(q, \tau) + \Delta l_x(q, \tau) \leqslant \frac{(\frac{3}{2}) - l_x(q, \tau)}{\tau},$$

we know there is a point $(\tilde{q}, \tilde{t})$, $\tilde{t} = \frac{1}{4}t_0$, such that $l_x(\tilde{q}, \tilde{\tau}) \leqslant \frac{3}{2}$, where $\tilde{\tau} = t_0 - \tilde{t}$. From the inequality (p. 197, Theorem 9.13 in [10]), $|\nabla l_x(q, \tau)|^2 \leqslant |\nabla l_x(q, \tau)|^2 + R(q, \tau) \leqslant (1 + 2n) l_x(q, \tau)/\tau$, we have

$$l_x(q, \tilde{\tau}) \leqslant \left(\frac{\sqrt{2n+1}\, d_{g(t_0 - \tilde{\tau})}(q, \tilde{q})}{2} + \sqrt{\frac{n}{2}}\right)^2,$$

so, for any $A < \infty$, we have $l_x(q, \tilde{\tau}) < C(A)$, when $(q, t_0 - \tilde{\tau}) \in B(\tilde{q}, t_0 - \tilde{\tau}, A)$, where $C(A)$ is a constant depend on A.

By the above injectivity radius estimate, we have a lower bound on $V(A)$, $\mathrm{Vol}_{g(t_0-\tilde{\tau})} B(\tilde{q}, A) \geqslant V(A)$.

By Perelman's no local collapsing theorem (p. 184, Theorem 8.1 in [10]), $|Rm(p,t)| < r^{-2}$, $(p,t) \in B(x, t_0, r) \times [t_0 - r^2, t_0]$, then $\mathrm{Vol}\, B(x, t_0, r) \geqslant \kappa r^n$. □

Remark 3.2. The detailed proof of Perelman's no local collapsing theorem has been given for balls of compact Ricci flow in Theorem 8.1 in p. 184 in [10] by using the monotonicity of reduced volume. However, one can see that the proof does not use of compactness of the flow. Since we have the injectivity radius estimate at the start and the reduced length is bounded on some ball, we have a low bound for the reduced volume at the start. By the monotonicity of reduced volume, we have low bound for the reduced volume of $g(t)$ for $t \in (0, T)$.

Lemma 3.3. *Fix $0 < \epsilon < 1$, then there is a positive constant $r > 0$ such that for any point (x_0, t_0) in the flow with $R(x_0, t_0) \geqslant r^{-2}$, (x_0, t_0) has a strong canonical $(C(\epsilon), \epsilon)$ neighborhood (p. 241, Definition 9.46 in [10]).*

Proof. By Theorem 2.5, we have, for any $t'_n < T$, $t'_n \to T$, the following curvature bound $|Rm(x,t)| < C(t'_n)$, $(x,t) \in M \times [0, t'_n]$. Set

$$A_n = \big\{(x,t) \in M \times [0, t'_n] \mid (x,t) \text{ does not have strong canonical neighborhood}\big\}.$$

Then there is also a uniform upper bound to curvature at any point in A_n, having, $R(x,t) < \tilde{C}(t'_n)$, $(x,t) \in A_n$. Hence, we can pick point $(x_n, t_n) \in A_n$, such that $R(x_n, t_n) > \frac{1}{2}\tilde{C}(t'_n)$, $t_n \leqslant t'_n$. We now have to prove that $\overline{\lim}_{n\to\infty} R(x_n, t_n) < \infty$.

By contradiction, we may assume that $Q_n = R(x_n, t_n) \to \infty$. By this, we know that $\forall (x,t) \in M \times [0, t_n]$, if $R(x,t) > 4R(x_n, t_n)$, (x,t) has a canonical neighborhood. Note that our flow is a 3-dimensional Ricci flow and the curvature is pinching toward positive (p. 251, Definition 10.1 in [10]). By Lemma 3.1, we can verify that the assumption of noncollapsing in Theorem 2.1 is true. Since $Q_n \to \infty$, $t_n \to T$, for any fixed $T_0 > 0$, we have $(t_n - T_0 Q_n^{-1}, t_n] \subset [0, t_n]$ for sufficiently large n. That is, the addition assumption of Theorem 2.3 is also satisfied.

By Theorem 2.3, $(M, Q_n g(t_n + \frac{t}{Q_n}), (x_n, t_n))$ converges to a limit flow $(M_\infty, g_\infty(t), (x_\infty, 0))$, which is a κ solution. So for sufficiently large n, (x_n, t_n) has a strong canonical neighborhood. This contradicts our assumption that none of the points (x_n, t_n) has a strong canonical neighborhood (p. 250, Corollary 9.84 in [10]). □

Proof of Theorem 1.1. Let us assume the Ricci flow blows up at finite time T. By Theorem 2.5, we know there is a limit metric $g(T)$ at infinity in the sense that $g(t)|_{M-K} \to g(T)|_{M-K}$, where K is a suitable compact set of M. Since the Ricci flow blows up at the time $t = T$, we have $\sup_{x\in M} Rm(x,t) \to \infty$, $t \to T$. (Otherwise, we can extend the flow with the curvature bounded, which contradicts the maximum of existence time.) So there is a point $p \in K$, such that the scalar curvature blows up at T, that is, $R(p,t) \to \infty$, $t \to T$. Then we can pick up a sequence $t_n \to T$ such that $Q_n = R(p, t_n) \to \infty$. By Lemma 3.3, the assumptions of Theorem 2.3 are satisfied. Hence, $(M, Q_n g(t_n + \frac{t}{Q_n}), (p, t_n))$ converges geometrically to a κ solution $(M_\infty, g_\infty(t), (x_\infty, 0))$, where $x_\infty = p$.

Recall that the asymptotic volume of κ solution is zero, that is,

$$\lim_{r\to\infty} \frac{\mathrm{vol}\, B(x_\infty, r)}{r^3} = 0.$$

Fixing any $\epsilon > 0$, there is a sufficient large r such that

$$\frac{\mathrm{vol}\, B(x_\infty, r)}{r^3} \leqslant \frac{\epsilon}{2}.$$

Since $(M, Q_n g(t_n + \frac{t}{Q_n}), (p, t_n))$ converges to $(M_\infty, g_\infty(t), (x_\infty, 0))$, for large n, we have

$$\frac{\mathrm{vol}\, B_{Q_n g(t_n)}((p, t_n), r)}{r^3} \leqslant \epsilon, \quad \text{and then} \quad \frac{\mathrm{vol}\, B_{g(t_n)}((p, t_n), r/Q_n^{1/2})}{(r/Q_n^{1/2})^3} \leqslant \epsilon.$$

On the other hand, there is a compact region $\Omega \subset M - K$, such that $g(t)|_\Omega \to g(T)|_\Omega$. Since $R > 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{d}\mu = -\int_\Omega R\, \mathrm{d}\mu \leqslant 0$, we have $\mathrm{Vol}_{g(t)}\, \Omega \geqslant \delta$, $t \in [0, T]$.

But Ω is compact, and we can find $\tilde{r} > 0$, such that $\Omega \subset B_{g(0)}(p, \tilde{r})$. Since $Ric \geqslant 0$, the distance function decreases with respect to t. $\Omega \subset B_{g(t)}(p, \tilde{r})$, $t \in [0, T)$, so $\operatorname{Vol} B_{g(t_n)}((p, t_n), \tilde{r}) \geqslant \delta$.

Since $Q_n \to \infty$, for sufficiently large n, $\tilde{r} > r/Q_n^{1/2}$. We now choose $\epsilon < \delta/\tilde{r}^3$. Using $Ric \geqslant 0$, and the Bishop–Gromov volume comparison theorem, we have that

$$\frac{\delta}{\tilde{r}^3} > \epsilon > \frac{\operatorname{Vol} B_{g(t_n)}((p, t_n), r/Q_n^{1/2})}{(r/Q_n^{1/2})^3} > \frac{\operatorname{Vol} B_{g(t_n)}((p, t_n), \tilde{r})}{\tilde{r}^3} \geqslant \frac{\delta}{\tilde{r}^3},$$

which is absurd. Therefore, the Ricci flow $(M, g(t))$ on $(M, g(0))$ is nonsingular at any finite time. In other words, $T = \infty$. This ends the proof of Theorem 1.1. □

Finally, we make some remarks about the proof above:

Remark 3.4. In the last part of the proof above, we consider the original Ricci flow, not the geometric limit flow.

Remark 3.5. Due to the positivity of curvature operator, the volume of a compact domain is decreasing along the Ricci flow. Since there is a low bound on volume of Ω at T, we have a low bound of the volume of Ω at any earlier time $t < T$.

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