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Poisson (co)homology of truncated polynomial algebras in two variables

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Abstract

We study the Poisson (co)homology of the algebra of truncated polynomials in two variables viewed as the semi-classical limit of a quantum complete intersection studied by Bergh and Erdmann. We show in particular that the Poisson cohomology ring of such a Poisson algebra is isomorphic to the Hochschild cohomology ring of the corresponding quantum complete intersection. ***To cite this article: S. Launois, L. Richard, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Cohomologie de Poisson des algèbres de polynômes tronqués en deux indéterminées. Nous étudions la cohomologie de Poisson d'une algèbre de polynômes tronqués en deux indéterminées vue comme la limite semi-classique des intersections complètes quantiques étudiées par Bergh et Erdmann. Nous montrons en particulier que l'anneau de cohomologie de Poisson de cette algèbre de Poisson est isomorphe à l'anneau de cohomologie de Hochschild de l'intersection complète quantique associée. ***Pour citer cet article : S. Launois, L. Richard, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

La cohomologie de Poisson d'une algèbre de Poisson, telle qu'introduite par Lichnerowicz dans [13], fournit des informations essentielles sur la structure de Poisson considérée. Ainsi la cohomologie en degré zéro correspond aux Casimirs, celle en degré un aux dérivations de Poisson modulo les dérivations hamiltonniennes, et les espaces de cohomologie en degrés deux et trois sont associés aux déformations de cette structure. Un problème classique en théorie des déformations algébriques consiste à comparer la (co)homologie de Poisson d'une algèbre de Poisson avec

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la (co)homologie de Hochschild de sa déformation. S'il est connu que ces homologies ont des comportements similaires pour des algèbres lisses (voir par exemple [7,9,10,12]), le cas singulier semble plus compliqué à appréhender. L'article [2] fournit de nombreux exemples où déjà les groupes des traces ne sont pas isomorphes. Même lorsque ces groupes coïncident comme c'est le cas pour les singularités de Klein dans [3], les résultats apparaissant dans [1] et [14] montrent que les espaces de (co)homologie de Poisson en plus haut degrés ne sont pas isomorphes aux espaces correspondant en (co)homologie de Hochschild.

Dans cette Note nous exhibons une algèbre de Poisson singulière pour laquelle l'anneau de cohomologie de Poisson est isomorphe comme algèbre graduée-commutative à l'anneau de cohomologie de Hochschild d'une déformation naturelle de cette algèbre de Poisson. Plus précisément, pour deux entiers $a, b \geqslant 2$, on considère l'algèbre de polynômes tronqués $\Lambda(a,b) := \mathbb{C}[X,Y]/(X^a, Y^b)$ munie du crochet de Poisson défini par $\{X, Y\} = XY$. Cette algèbre est la limite semi-classique de l'intersection complète quantique $\Lambda_q(a,b)$ étudiée par Bergh et Erdmann dans [5]. Les propriétés homologiques de ces algèbres non commutatives de dimension finie ont été l'objet de nombreuses études récentes (voir par exemple [8,4–6]). En particulier l'algèbre extérieure quantique (dans le cas générique), qui correspond au cas $a = b = 2$, fournit un contre-exemple à la question de Happel : il est démontré dans [8] que $\Lambda_q(2,2)$ est de dimension globale infinie alors que ses groupes de cohomologie de Hochschild sont nuls en degré supérieur à 3. L'anneau de cohomologie de Hochschild de $\Lambda_q(a,b)$ a été décrit récemment dans [5] lorsque q n'est pas racine de l'unité. C'est une algèbre graduée de dimension cinq, isomorphe au produit fibré $\mathcal{F} := \mathbb{C}[U]/(U^2) \times_{\mathbb{C}} \mathbb{C}\langle V, W\rangle/(V^2, VW + WV, W^2)$, avec U en degré zéro et V et W en degré un. Dans cette Note nous obtenons la même description pour l'anneau de cohomologie de Poisson de $\Lambda(a,b)$, conduisant au

Théorème 0.1. *Pour tous $a, b \geqslant 2$, on a les isomorphismes suivants d'algèbres graduées-commutatives : $HP^*(\Lambda(a,b)) \simeq \mathcal{F} \simeq HH^*(\Lambda_q(a,b))$, où $HP^*(\Lambda(a,b))$ est munie du cup-produit de Poisson induit par le produit extérieur sur les multi-dérivations antisymétriques.*

Remarquons que $\Lambda(a,b)$ n'est pas unimodulaire (voir par exemple [11] pour un exposé récent sur la classe modulaire d'une variété de Poisson). En effet, il n'y a pas de dualité de Poincaré entre $HP^k(\Lambda(a,b))$ et $HP_{2-k}(\Lambda(a,b))$. Par exemple, $HP^2(\Lambda(a,b))$ est un espace de dimension un alors que $HP_0(\Lambda(a,b))$ a pour dimension $a + b - 1$. On ne peut d'ailleurs pas espérer une dualité tordue semblable à celle décrite dans [12] pour un espace affine (voir le paragraphe qui suit le Lemme 3.1). Toutefois, l'automorphisme de Nakayama ν provenant de la structure de Frobenius de l'intersection complète quantique $\Lambda_q(a,b)$ (voir [5]) permet de définir un module de Poisson M_ν tel que

Théorème 0.2. *$HP^k(\Lambda(a,b)) \simeq HP_k(\Lambda(a,b), M_\nu)$ pour tout entier naturel k.*

1. Introduction

Given a Poisson algebra, its Poisson cohomology—as introduced by Lichnerowicz in [13]—provides important informations about the Poisson structure (the Casimir elements are reflected by the degree zero cohomology, Poisson derivations modulo Hamiltonian derivations by the degree one,...). It also plays a crucial role in the study of deformations of the Poisson structure. A classical problem in algebraic deformation is to compare the Poisson (co)homology of a Poisson algebra with the Hochschild (co)homology of its deformation. Although these homologies are known to behave similarly in smooth cases (see e.g. [7,9,10,12]), the singular case seems more complicated to deal with. There are many examples in which the trace groups already do not match in [2]. And going through the results of [3,1,14] one sees that even though the dimension of the homology spaces in degree zero match, it might not be the case in higher degree.

Our aim in this Note is to provide an example of a singular Poisson algebra such that its Poisson cohomology ring is isomorphic as a graded commutative algebra to the Hochschild cohomology ring of a natural deformation. Namely for two integers $a, b \geqslant 2$ we consider the truncated polynomial algebra $\Lambda(a,b) := \mathbb{C}[X,Y]/(X^a, Y^b)$ with the Poisson bracket given by $\{X, Y\} = XY$. This algebra is the semi-classical limit of the quantum complete intersection $\Lambda_q(a,b)$ studied by Bergh and Erdmann in [5]. The homological properties of this class of noncommutative finite-dimensional algebras have been extensively studied recently (see for instance [8,4–6]). In particular it is proved in [8] that the generic quantum exterior algebra (corresponding to the case $a = b = 2$) provides a counter-example to Happel's question. That is, the global dimension of $\Lambda_q(2,2)$ is infinite whereas its Hochschild cohomology groups vanish in degree greater than 2. Recently the Hochschild cohomology ring of $\Lambda_q(a,b)$ has been described in [5] when q is not a root of unity. It is a five-dimensional graded algebra isomorphic to the fibre product $\mathcal{F} := \mathbb{C}[U]/(U^2) \times_{\mathbb{C}} \mathbb{C}\langle V, W\rangle/(V^2, VW + WV, W^2)$, with U in degree zero and V and W in degree one. In this Note we obtain the same description for the Poisson cohomology ring of $\Lambda(a,b)$ leading to

Theorem 1.1. *For all $a, b \geqslant 2$ one has $HP^*(\Lambda(a,b)) \simeq \mathcal{F} \simeq HH^*(\Lambda_q(a,b))$ as graded commutative algebras, where $HP^*(\Lambda(a,b))$ is endowed with the Poisson cup product induced by the exterior product on skew-symmetric multi-derivations.*

Note that $\Lambda(a,b)$ is not unimodular (see for instance [11] for a recent survey on the modular class of a Poisson manifold). Indeed, there is no Poincaré duality between $HP^k(\Lambda(a,b))$ and $HP_{2-k}(\Lambda(a,b))$. For instance $HP^2(\Lambda(a,b))$ has dimension 1 whereas $HP_0(\Lambda(a,b))$ has dimension $a+b-1$. One cannot even expect a twisted Poincaré duality as in [12] (see the paragraph below Lemma 3.1). However, the Nakayama automorphism ν coming from the Frobenius structure of the quantum complete intersection $\Lambda_q(a,b)$ (see [5]) allows us to construct a Poisson module M_ν such that

Theorem 1.2. *$HP^k(\Lambda(a,b)) \simeq HP_k(\Lambda(a,b), M_\nu)$ for all nonnegative integers k.*

2. Poisson cohomology of truncated polynomials

The algebra $\Lambda(a,b)$ has dimension ab, with basis $\{e_{ij} := X^iY^j \mid 0 \leqslant i \leqslant a-1,\ 0 \leqslant j \leqslant b-1\}$. One easily checks that the following formulas hold in $\Lambda(a,b)$ for all i, j:

$$\{X^iY^j, X\} = -jX^{i+1}Y^j; \qquad \{X^iY^j, Y\} = iX^iY^{j+1}. \tag{1}$$

Recall that the complex computing the Poisson cohomology is (χ^k, δ_k), with χ^k the space of skew-symmetric k-derivations of $\Lambda(a,b)$. Interestingly, these spaces vanish for $k \geqslant 3$, since $\Lambda(a,b)$ is 2-generated. The Poisson coboundary operator $\delta_k : \chi^k \to \chi^{k+1}$ is defined by

$$\delta_k(P)(f_0, \dots, f_k) := \sum_{i=0}^{k} (-1)^i \{f_i, P(f_0, \dots, \hat{f_i}, \dots, f_k)\} + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} P(\{f_i, f_j\}, f_0, \dots, \hat{f_i}, \dots, \hat{f_j}, \dots, f_k)$$

for all $P \in \chi^k$ and $f_0, \dots, f_k \in \Lambda(a,b)$. It is easy to check that $\delta_k(P)$ belongs indeed to χ^{k+1} and that $\delta_{k+1} \circ \delta_k = 0$. The kth Poisson cohomology space of $\Lambda(a,b)$, denoted by $HP^k(\Lambda(a,b))$, is the kth cohomology space of this complex. The Poisson cohomology ring is the space $HP^*(\Lambda(a,b)) := \bigoplus_{k=0}^{\infty} HP^k(\Lambda(a,b))$. Endowed with the cup product induced by the exterior product on $\chi^* := \bigoplus_k \chi^k$ it becomes a graded commutative algebra. We start by describing the spaces $HP^k(\Lambda(a,b))$ for $k = 0, 1, 2$ as it is clear that $HP^k(\Lambda(a,b)) = 0$ for $k \geqslant 3$.

Let $\lambda = \sum \lambda_{ij} e_{ij} \in \Lambda(a,b)$, and assume it satisfies $\{\lambda, X\} = \{\lambda, Y\} = 0$. Then Eqs. (1) lead successively to $\lambda_{ij} = 0$ for all $i \neq a-1$ and all $j \neq 0$, and then to $\lambda_{ij} = 0$ for all $i \neq 0$ and $j \neq b-1$. Hence

Proposition 2.1. *The Poisson centre $HP^0(\Lambda(a,b))$ is equal to $\mathbb{C} \oplus \mathbb{C}X^{a-1}Y^{b-1} = \mathbb{C}e_{0,0} \oplus \mathbb{C}e_{a-1,b-1}$.*

Any derivation $d \in \chi^1$ is uniquely determined by the values of $d(X) = \sum \lambda_{ij} e_{ij}$ and $d(Y) = \sum \lambda'_{ij} e_{ij}$. Moreover, d must satisfy the relations $d(X^a) = d(Y^b) = 0$, that is $X^{a-1}d(X) = Y^{b-1}d(Y) = 0$, since d is a derivation. From that one easily deduces that $\lambda_{0j} = \lambda'_{i0} = 0$ for all i, j. Hence the space $\chi^1 = \mathrm{Der}(A, A)$ has basis $\{d_{ij}\} \cup \{d'_{ij}\}$, where:

(i) for $1 \leqslant i < a$ and $0 \leqslant j < b$, the derivation d_{ij} is defined by $d_{ij}(X) = X^iY^j$ and $d_{ij}(Y) = 0$;
(ii) for $0 \leqslant i < a$ and $1 \leqslant j < b$, the derivation d'_{ij} is defined by $d'_{ij}(X) = 0$ and $d'_{ij}(Y) = X^iY^j$.

In particular, $\dim(\chi^1) = b(a-1) + a(b-1)$. Let $d = \sum_{i\neq 0} \alpha_{i,j} d_{i,j} + \sum_{j\neq 0} \beta_{i,j} d'_{i,j} \in \chi^1$. Then $d \in \operatorname{Ker}\delta_1$ if and only if it satisfies $d(\{X,Y\}) = \{d(X),Y\} + \{X,d(Y)\}$, that is:

$$\sum_{j\neq 0} \beta_{i,j} X^{i+1}Y^j + \sum_{i\neq 0} \alpha_{i,j} X^i Y^{j+1} = \sum_{i\neq 0} i\alpha_{i,j} X^i Y^{j+1} + \sum_{j\neq 0} j\beta_{i,j} X^{i+1}Y^j.$$

Identifying the coefficients in front of X^iY^j leads to

$$d \in \operatorname{Ker}\delta_1 \iff (1-j)\beta_{i-1,j} + (1-i)\alpha_{i,j-1} = 0 \quad \forall 1 \leqslant i \leqslant a-1,\ 1 \leqslant j \leqslant b-1. \tag{2}$$

Proposition 2.2. $HP^1(\Lambda(a,b)) = \mathbb{C}d_{1,0} \oplus \mathbb{C}d'_{0,1}$.

Proof. Let $d = \sum_{i\neq 0} \alpha_{i,j} d_{i,j} + \sum_{j\neq 0} \beta_{i,j} d'_{i,j} \in \operatorname{Ker}\delta_1$. Set $\lambda := \sum_{j\neq 0} \frac{\alpha_{i+1,j}}{j} X^iY^j \in \Lambda(a,b)$. From (1) one deduces that $d_1 = d + \{\lambda, -\}$ is a Poisson derivation satisfying $d_1(X) = \sum_{i\geqslant 1} \alpha_{i,0} X^i$. Then one deduces from (2) that $\alpha_{i,0} = 0$ for all $i \neq 1$, that is $d_1(X) = \alpha_{1,0}X$, and $d_1(Y) = \sum_{i=0}^{a-2} \beta_{i,1} X^iY + \sum_{j=1}^{b-1} \beta_{a-1,j} X^{a-1}Y^j$. Now set $\mu := \sum_{i=1}^{a-2} \frac{\beta_{i,1}}{i} X^i + \sum_{j=1}^{b-1} \frac{\beta_{a-1,j}}{a-1} X^{a-1}Y^{j-1}$ and $d_2 := d_1 - \{\mu, -\}$. From the construction we get that $d_2 = \alpha_{1,0} d_{1,0} + \beta_{0,1} d'_{0,1}$, so that the images of $d_{1,0}$ and $d'_{0,1}$ span $HP^1(\Lambda(a,b))$. One deduces from (1) that they actually form a basis of $HP^1(\Lambda(a,b))$. □

The complex computing the Poisson cohomology is vanishing after χ^2, so we use the Euler–Poincaré principle to compute the dimension of $HP^2(\Lambda(a,b))$. First note that a skew-symmetric derivation $f \in \chi^2$ is determined by $f(X\wedge Y) = \sum c_{ij} e_{ij}$, with $aX^{a-1} f(X\wedge Y) = bY^{b-1} f(X\wedge Y) = 0$, so that $c_{0j} = c_{i0} = 0$ for all i, j. Thus χ^2 has dimension $(a-1)(b-1)$.

Proposition 2.3. *$HP^2(\Lambda(a,b)) = \mathbb{C}f_{1,1}$, with $f_{1,1} : X\wedge Y \mapsto XY$.*

Proof. We first prove that $HP^2(\Lambda(a,b))$ has dimension 1. From the Euler–Poincaré principle we get

$$\begin{aligned}\dim\bigl(HP^2\bigl(\Lambda(a,b)\bigr)\bigr) &= \dim\chi^2 - \operatorname{rg}\delta_1 = (a-1)(b-1) - \bigl(\dim\chi^1 - \dim\operatorname{Ker}\delta_1\bigr)\\ &= (a-1)(b-1) - \bigl[a(b-1) + b(a-1) - \bigl(\dim HP^1\bigl(\Lambda(a,b)\bigr) + \operatorname{rg}\delta_0\bigr)\bigr]\\ &= 1 - ab + \bigl(2 + \dim\Lambda(a,b) - \dim\operatorname{Ker}\delta_0\bigr) = 3 - ab + ab - 2 = 1.\end{aligned}$$

Now all that remains is to check that $X\wedge Y \mapsto XY$ is not a Poisson coboundary. Any $P \in \chi^1$ satisfies $\delta_1(P)(X\wedge Y) = \{X, P(Y)\} - \{Y, P(X)\} - P(XY)$. Moreover one has $P(XY) = P(X)Y + P(Y)X$, and it results straight from formulas (1) that one cannot have $\delta_1(P)(X\wedge Y) = XY$. □

Proof of Theorem 1.1. We have already proved that the graded commutative algebra $HP^*(\Lambda(a,b))$ is five-dimensional with basis $(e_{0,0}, e_{a-1,b-1}, d_{1,0}, d'_{0,1}, f_{1,1})$ with degree respectively $(0,0,1,1,2)$. One can easily check that $e_{a-1,b-1}$ annihilates $d_{1,0}$, $d'_{0,1}$ and $f_{1,1}$, and that $d_{1,0} \smile d'_{0,1} = f_{1,1}$. Thus we have $HP^*(\Lambda(a,b)) \simeq \mathcal{F}$, and one concludes using [5, Theorem 3.3]. □

3. (Twisted) Poisson homology

Let M be a right Poisson module over $\Lambda(a,b)$, i.e. a $\mathbb{C}$-vector space M endowed with two bilinear maps . and $\{\cdot,\cdot\}_M$ such that

(i) $(M, \cdot)$ is a (right) module over the commutative algebra R,
(ii) $(M, \{\cdot,\cdot\}_M)$ is a (right) module over the Lie algebra $(R, \{\cdot,\cdot\})$,
(iii) $x.\{a,b\} = \{x.a, b\}_M - \{x.b, a\}_M$ for all $a, b \in R$ and $x \in M$,
(iv) $\{x, ab\}_M = \{x,a\}_M.b + \{x,b\}_M.a$ for all $a, b \in R$ and $x \in M$.

Then one defines a chain complex on the $\Lambda(a,b)$-modules $M \otimes_{\Lambda(a,b)} \Omega^k$, where Ω^k denotes the so-called Kähler differential k-forms of $\Lambda(a,b)$, as follows. The boundary operator $\partial_k : M \otimes_{\Lambda(a,b)} \Omega^k \to M \otimes_{\Lambda(a,b)} \Omega^{k-1}$ is defined by

$$\partial_k(m \otimes \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_k) = \sum_{i=1}^{k} (-1)^{i+1} \{m, a_i\}_M \otimes \mathrm{d}a_1 \wedge \cdots \wedge \widehat{\mathrm{d}a_i} \wedge \cdots \wedge \mathrm{d}a_k + \sum_{1 \leqslant i < j \leqslant k} (-1)^{i+j} m \otimes \mathrm{d}\{a_i, a_j\} \wedge \mathrm{d}a_1 \wedge \cdots \wedge \widehat{\mathrm{d}a_i} \wedge \cdots \wedge \widehat{\mathrm{d}a_j} \wedge \cdots \wedge \mathrm{d}a_k,$$

where we have removed the expressions under the hats in the previous sums and d denotes the exterior differential. The homology of this complex is denoted by $HP_*(\Lambda(a,b), M)$ and called the Poisson homology of the Poisson algebra $\Lambda(a,b)$ with values in the Poisson module M. Before computing the Poisson homology spaces for a specific module, we describe the spaces Ω^k. By definition Ω^k is a $\Lambda(a,b)$-module generated by the wedge products of length k of the 1-differential forms $\mathrm{d}X$, $\mathrm{d}Y$. In particular $\Omega^k = 0$ for $k \geqslant 3$. In the remaining cases the torsion coming from the relations $X^a = Y^b = 0$ leads to the following spaces:

$$\Omega^0 = \Lambda(a,b), \quad \Omega^1 = \bigoplus_{\substack{0 \leqslant i \leqslant a-2 \\ 0 \leqslant j \leqslant b-1}} \mathbb{C} X^i Y^j \, \mathrm{d}X \oplus \bigoplus_{\substack{0 \leqslant i \leqslant a-1 \\ 0 \leqslant j \leqslant b-2}} \mathbb{C} X^i Y^j \, \mathrm{d}Y,$$

and

$$\Omega^2 = \bigoplus_{\substack{0 \leqslant i \leqslant a-2 \\ 0 \leqslant j \leqslant b-2}} \mathbb{C} X^i Y^j \, \mathrm{d}X \wedge \mathrm{d}Y.$$

Their dimensions are respectively ab, $(a-1)b + b(a-1)$, and $(a-1)(b-1)$. As the bracket $\{\lambda, \mu\}$ belongs to the ideal generated by XY for any $\lambda, \mu \in \Lambda(a,b)$, one easily checks that:

Lemma 3.1. *$HP_0(\Lambda(a,b))$ has dimension $a+b-1$.*

As $a+b-1 \geqslant 3$ for any $a, b \geqslant 2$, and $HP^2(\Lambda(a,b))$ has dimension 1, this lemma shows that there is no Poincaré duality between $HP^k(\Lambda(a,b))$ and $HP_{2-k}(\Lambda(a,b))$.

We may ask now if there is a twisted duality similar to the one obtained in [12]. The Poisson algebra $\Lambda(a,b)$ is the semi-classical limit of the quantum complete intersection $\Lambda_q(a,b)$ which is the $\mathbb{C}$-algebra generated by x, y with relations $xy = qyx$, $x^a = 0$, $y^b = 0$ (see [5]). Any diagonal automorphism σ of $\Lambda_q(a,b)$ defined by $\sigma(x) = q^\alpha x$ and $\sigma(y) = q^\beta y$ gives rise to a Poisson module M_σ of $\Lambda(a,b)$ via a semi-classical limit process (see [12, Section 3.1] for details). As a vector space M_σ is equal to $\Lambda(a,b)$, and the external Poisson bracket is defined by:

$$\left\{X^i Y^j, X\right\}_{M_\sigma} = -(j+\alpha) X^{i+1} Y^j \quad \text{and} \quad \left\{X^i Y^j, Y\right\}_{M_\sigma} = (i-\beta) X^i Y^{j+1}. \tag{3}$$

From these formulas one easily deduces that if $\alpha < -b+1$ and $\beta > a-1$, then $HP_0(\Lambda(a,b), M_\sigma) = \mathbb{C}$. So one might be tempted to use such a Poisson module to restore a twisted Poincaré duality between $HP^k(\Lambda(a,b))$ and $HP_{2-k}(\Lambda(a,b), M_\sigma)$ as in [12, Theorem 3.4.2]. Unfortunately, these hypotheses on α and β also lead to $HP_2(\Lambda(a,b), M_\sigma) = 0$, so that $HP_2(\Lambda(a,b), M_\sigma)$ is not isomorphic to $HP^0(\Lambda(a,b))$.

The Nakayama automorphism ν of $\Lambda_q(a,b)$ coming from the Frobenius algebra structure of $\Lambda_q(a,b)$ is defined by $\nu(x) = q^{1-b} x$ and $\nu(y) = q^{a-1} y$. This automorphism was used in [5, Section 3] to link the twisted Hochschild homology and the Hochschild cohomology of $\Lambda_q(a,b)$ in each degree. We end this Note by computing the dimensions of the Poisson homology spaces of $\Lambda(a,b)$ with value in the Poisson module M_ν corresponding to the Nakayama automorphism ν.

Proposition 3.2. *The twisted Poisson homology spaces $HP_k(\Lambda(a,b), M_\nu)$ have dimension $2, 2, 1$ for $k = 0, 1, 2$ respectively, and are null if $k \geqslant 3$.*

Proof. From $\partial_1(X^i Y^j \otimes \mathrm{d}X) = \{X^i Y^j, X\}_{M_\nu} = -(j-b+1) X^{i+1} Y^j$ and $\partial_1(X^i Y^j \otimes \mathrm{d}Y) = \{X^i Y^j, Y\}_{M_\nu} = (i-a+1) X^i Y^{j+1}$ one easily gets that $HP_0(\Lambda(a,b), M_\nu) = \Lambda(a,b)/\operatorname{Im} \partial_1$ is generated as a $\mathbb{C}$-vector space by the classes of 1 and $X^{a-1} Y^{b-1}$ modulo $\operatorname{Im} \partial_1$.

We compute $\partial_2(X^iY^j \otimes \mathrm{d}X \wedge \mathrm{d}Y) = \{X^iY^j, X\}_{M_\nu} \otimes \mathrm{d}Y - \{X^iY^j, Y\}_{M_\nu} \otimes \mathrm{d}X - X^iY^j \otimes (X\,\mathrm{d}Y + Y\,\mathrm{d}X) = -(j-b+2)X^{i+1}Y^j \otimes \mathrm{d}Y - (i-a+2)X^iY^{j+1} \otimes \mathrm{d}X$, from which one easily sees that $\operatorname{Ker}\partial_2 = HP_2(\Lambda(a,b), M_\nu)$ is the $\mathbb{C}$-vector space generated by $X^{a-2}Y^{b-2} \otimes \mathrm{d}X \wedge \mathrm{d}Y$.

We use the Euler–Poincaré principle to conclude. There is only one dimension missing, all the other ones have the same value as for the Poisson cohomology of $\Lambda(a,b)$, so we end up with the same value for $\dim HP_1(\Lambda(a,b), M_\nu)$, that is 2. □

We deduce from Proposition 3.2 and Theorem 1.1 a duality similar to the one appearing in [5], that is $HP^k(\Lambda(a,b)) \simeq HP_k(\Lambda(a,b), M_\nu)$ for all nonnegative integers k as claimed in Theorem 1.2.

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