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Homogeneous Einstein–Randers spaces of negative Ricci curvature ☆

Shaoqiang Deng, Zixin Hou

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China

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Abstract

We prove that a homogeneous Einstein–Randers space with negative Ricci curvature must be Riemannian. ***To cite this article: S. Deng, Z. Hou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Espaces Einstein–Randers homogènes avec courbure de Ricci négative. Nous prouvons que l'espace Einstein–Randers homogéne avec courbure de Ricci négative doit être Riemannian. ***Pour citer cet article : S. Deng, Z. Hou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

In Finsler geometry, the flag curvature and Ricci scalar are the analogues of sectional curvature and Ricci curvature in Riemannian geometry. They play important roles in almost all aspects of Finsler geometry. For example, the Finslerian version of the Cartan–Hadamard theorem asserts that a connected simply connected Finsler space with non-positive flag curvature must be diffeomorphic to a euclidean space. The Bonnet–Myers theorem asserts that a complete Finsler space whose Ricci scalar is bounded from below by a positive number must be compact. It is therefore interesting and important to study how flag curvature and Ricci scalar will influence the properties of a Finsler space.

The purpose of this short Note is to prove the following interesting result:

Main Theorem. *A homogeneous Einstein–Randers space with negative Ricci curvature is Riemannian.*

We remark here that Bao and Robles have proved in [3] that a **compact** Einstein–Randers space with negative Ricci curvature must be Riemannian, without the homogeneous condition. However, their proof does not apply to the non-compact case. As a fact we must point out that a homogeneous Einstein Finsler space (not necessary of the Randers type) with negative Ricci curvature may very likely be non-compact. For example, a homogeneous Einstein Riemannian manifold with negative Ricci curvature must be non-compact, see [5, p. 190].

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E-mail address: dengsq@nankai.edu.cn (S. Deng).

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2. Characterization of Einstein–Randers spaces

A Randers metric is a Finsler metric built from a Riemannian metric α and a 1-form β whose length with respect to α is everywhere less than 1. The metric is written as $F = \alpha + \beta$. However, sometimes this defining form is not convenient. There is another version of such metrics which is extremely convenient in handling Einstein–Randers metrics and metrics of constant flag curvature. This is the so called navigation data introduced by Z. Shen in [8]. Precisely, Shen proved that there is a Riemannian metric h and a vector field W with $h(W, W) < 1$ everywhere, such that

$$F(y) = \frac{\sqrt{[h(W,y)]^2 + |y|^2\lambda}}{\lambda} - \frac{h(W,y)}{\lambda},$$

where $\lambda = 1 - |W|^2$ and the length function is taken with respect to h (see [3]). We call the pair (h, W) the navigation data of the corresponding Randers metric F.

Einstein–Randers metrics can be characterized in a very elegant way using the navigation data:

Theorem 2.1. *(See [3].) Let (M, F) be a Randers space with the navigation data (h, W). Then F is Einstein with the Ricci curvature $Ric(x, y) = (n-1)K(x)$ if and only if there exists a real number σ such that the following two condition holds*:

(1) *h is Einstein with Ricci scalar $(n-1)(K(x) + \frac{1}{16}\sigma^2)$.*
(2) *W is an infinitesimal homothety of h, namely, $\mathcal{L}_W h = -\sigma h$. Furthermore, σ must vanish whenever h is not Ricci-flat.*

3. Proof of the Main Theorem

Let (M, F) be a Randers space with navigation data (h, W) and suppose the Lie group G has a smooth effective action on M. Then it is easily seen that F is invariant under the action of G if and only if both h and W are invariant under the action of G. If the action of G on M is transitive, then M can be written as a coset space G/H, where H is the isotropy subgroup of G at a fixed point $x \in M$. Since the isotropy subgroup of the full group of isometries of (M, F) at x must be compact [6], we can assume that H is a compact subgroup of G. In this case, the coset space is reductive, in the sense that there is a decomposition of the Lie algebra:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum of subspaces)},$$

where $\mathfrak{g}$, resp. $\mathfrak{h}$, is the Lie algebra of G, resp. H and $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ satisfying $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}$. Through this decomposition, we can identify the tangent space $T_{eH}(G/H)$ of G/H at the origin eH with $\mathfrak{m}$, through the mapping $X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)H)|_{t=0}$, $X \in \mathfrak{m}$. Under this identification, the isotropy representation of H at $T_{eH}(G/H)$ corresponds to the adjoint representation on H at $\mathfrak{m}$.

If W is a G-invariant vector field on G/H, then the restriction of W to $T_{eH}(G/H)$ must be invariant under H. Under the above identification, W corresponds to a vector $X \in \mathfrak{m}$ such that

$$Ad(h)(X) = X, \quad \forall h \in H. \tag{1}$$

On the other hand, if $X \in \mathfrak{m}$ satisfies (1), then we can define a vector field W on G/H by $W|_{gH} = \frac{\mathrm{d}}{\mathrm{d}t}(g\exp(tX)H)|_{t=0}$. By (1), one can easily check that W is well-defined and invariant under G. Therefore, G-invariant vector fields on G/H are one-to-one corresponding to vectors in $\mathfrak{m}$ satisfying (1). This correspondence has been described in [7]. Moreover, (1) implies that

$$[Y, X] = 0, \quad \forall Y \in \mathfrak{h}. \tag{2}$$

Now we consider the Killing vector fields of invariant Riemannian metric on G/H. If $X \in \mathfrak{m}$ satisfies (1) and W is the corresponding G-invariant vector field, then W is a Killing vector field if and only if the one-parameter transformation group

$$\varphi_t : G/H \to G/H, \qquad gH \mapsto g\exp(tX)H, \quad t \in \mathbb{R}$$

consists of isometries of h. In particular, for $Z_1, Z_2 \in \mathfrak{m}$, we have

$$h(Z_1, Z_2) = h\big(\mathrm{d}\varphi_t(Z_1), \mathrm{d}\varphi_t(Z_2)\big). \tag{3}$$

Now we compute $\mathrm{d}(\varphi_t)(Z_i)$. Since Z_i is the initial vector of the curve $(\exp(sZ_i))H$ and

$$\varphi_t\big(\exp(sZ_i)H\big) = \exp(sZ_i)\exp(tX)H,$$

we have

$$\mathrm{d}(\varphi_t)(Z_i) = \frac{\mathrm{d}}{\mathrm{d}s}\big(\exp(sZ_i)\exp(tX)H\big)\big|_{s=0}.$$

Now

$$\exp(sZ_i)\exp(tX)H = \exp(tX)\exp(-tX)\exp(sZ_i)\exp(tX)H. \tag{4}$$

Since

$$\exp(-tX)\exp(sZ_i)\exp(tX) = \exp\big(Ad\big(\exp(tX)\big)(sZ_i)\big) = \exp\big(se^{ad(tX)}(Z_i)\big),$$

taking the derivative with respect to s in (4) we get

$$\mathrm{d}(\varphi_t)(Z_i) = \frac{\mathrm{d}}{\mathrm{d}s}\big[\exp(tX)\exp\big(se^{ad(tX)}(Z_i)\big)H\big]\big|_{s=0} = \mathrm{d}L_{\exp tX}\big[e^{ad(tX)}(Z_i)\big]_{\mathfrak{m}}, \tag{5}$$

where $L_{\exp(tX)}$ is the transformation of G/H define by $gH \to \exp(tX)gH$ and for $Y \in \mathfrak{g}$, $Y_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of Y with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Since h is G-invariant, $L_{\exp(tX)}$ are all isometries. Therefore (3) and (5) imply that

$$h(Z_1, Z_2) = h\big(\big[e^{ad(tX)}(Z_1)\big]_{\mathfrak{m}}, \big[e^{ad(tX)}(Z_2)\big]_{\mathfrak{m}}\big), \quad \forall t \in \mathbb{R}.$$

Taking the derivative with respect to t and considering the value at $t = 0$ we get:

$$h\big([X, Z_1]_{\mathfrak{m}}, Z_2\big) + h\big(Z_1, [X, Z_2]_{\mathfrak{m}}\big) = 0, \quad \forall Z_1, Z_2 \in \mathfrak{m}. \tag{6}$$

Conversely, if (6) holds, then a backward argument of the above implies that W is a Killing vector field of h.

In summarizing, we have proved

Proposition 3.1. *Suppose G is a connected Lie group and H is a closed subgroup such that G/H is a reductive homogeneous space with a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let h be a G-invariant Riemannian metric on G/H and suppose $X \in \mathfrak{m}$ satisfies* (1). *The corresponding invariant vector filed W on G/H is a Killing vector field with respect to h if and only if X satisfies* (6).

Combining Theorem 2.1 and Proposition 3.1, we get the characterization of homogeneous Einstein–Randers spaces.

Theorem 3.2. *Let G be a connected Lie group and H be a closed subgroup of G such that G/H is a reductive homogeneous space with a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Suppose h is a G-invariant Riemannian metric on G/H and $X \in \mathfrak{m}$ satisfies* (1) *with $h(X, X) < 1$. Let W be the corresponding vector field on G/H. Then the Randers metric F with navigation data (h, W) is Einstein with Ricci constant K if and only if h is Einstein with Ricci constant K and X satisfies* (6).

Proof. The "if" part is obvious. Now we prove the "only if" part. Assume that F is Einstein with Ricci constant K. If $K \geqslant 0$, then by Theorem 2.1, we see that h must be Einstein with Ricci constant K and the constant σ there has to be 0. Hence W must be a Killing vector field. If $K < 0$, then we have the following two cases:

I. h is Ricci flat. If W is not a Killing vector field, then $\sigma \neq 0$. Hence the Ricci constant of F is $-\frac{1}{16}\sigma^2 < 0$. Now a theorem of D.V. Alekseevskii and B.N. Kinmel'fel'd [2] asserts that a Ricci-flat homogeneous Riemannian manifold must be locally euclidean. Thus h is of constant sectional curvature 0. But then by D. Bao, C. Robles and Z. Shen's result [4] F must be of constant flag curvature $-\frac{1}{16}\sigma^2$. Since a homogeneous Finsler space must be complete and its

Cartan tensor is invariant, the Akbar-Zadeh theorem [1] then implies that F must be Riemannian. Hence $W = 0$. This is a contradiction. Hence W must be a Killing vector field.

II. h is not Ricci flat. Then Theorem 2.1 assures that $\sigma = 0$ hence W must be a Killing vector field.

This completes the proof of the theorem. □

Proof of the Main Theorem. Assume that (M, F) is a homogeneous Einstein–Randers metric with negative Ricci curvature K. Then we can write M as a coset space G/H of a Lie group G with G/H reductive. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebra of G and $\mathfrak{h}$, respectively and let $\mathfrak{m}$, h, W, X be as above. Then by Theorem 3.2, h is Einstein with Ricci constant K and X satisfies (6). Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis of $\mathfrak{m}$ with respect to h and suppose U is the bilinear mapping from $\mathfrak{m} \times \mathfrak{m}$ to $\mathfrak{m}$ defined by

$$2h\big(U(Z_1, Z_2), Z_3\big) = h\big([Z_3, Z_1]_{\mathfrak{m}}, Z_2\big) + h\big(Z_1, [Z_3, Z_2]_{\mathfrak{m}}\big), \quad Z_1, Z_2, Z_3 \in \mathfrak{m}.$$

Denote $Z = \sum_{i=1}^{n} U(X_i, X_i)$. Then the Ricci curvature of h can be expressed as (see [5, p. 184]):

$$\begin{aligned} r(Y, Y) = {} & -\frac{1}{2}\sum_{i=1}^{n}\big|[Y, X_i]_{\mathfrak{m}}\big|^2 - \frac{1}{2}\sum_{i=1}^{n} h\big(\big[Y, [Y, X_i]_{\mathfrak{m}}\big]_{\mathfrak{m}}, X_i\big) - \sum_{i=1}^{n} h\big(\big[Y, [Y, X_i]_{\mathfrak{h}}\big]_{\mathfrak{m}}, X_i\big) \\ & + \frac{1}{4}\sum_{i,j=1}^{n} h\big([X_i, X_j]_{\mathfrak{m}}, Y\big)^2 - h\big([Z, Y]_{\mathfrak{m}}, Y\big), \quad Y \in \mathfrak{m}. \end{aligned}$$

Now we consider the value at X. By (6) we have

$$h\big(\big[X, [X, X_i]_{\mathfrak{m}}\big]_{\mathfrak{m}}, X_i\big) = -h\big([X, X_i]_{\mathfrak{m}}, [X, X_i]_{\mathfrak{m}}\big) = -\big|[X, X_i]_{\mathfrak{m}}\big|^2.$$

Therefore the first term and the second term of $r(X, X)$ sum to 0. By (2), the third term is equal to 0. Now by (6)

$$h\big([Z, X]_{\mathfrak{m}}, X\big) = -h\big(Z, [X, X]_{\mathfrak{m}}\big) = 0.$$

Therefore we have

$$r(X, X) = \frac{1}{4}\sum_{i,j=1}^{n} h\big([X_i, X_j]_{\mathfrak{m}}, X\big)^2.$$

The Einstein condition thus implies that

$$\frac{1}{4}\sum_{i,j=1}^{n} h\big([X_i, X_j]_{\mathfrak{m}}, X\big)^2 = K h(X, X).$$

The assumption $K < 0$ then forces $X = 0$. Thus F must be Riemannian and the theorem is proved. □

Finally, we conclude the article with the following:

Conjecture. *A homogeneous Einstein Finsler space with negative Ricci curvature is Riemannian.*

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