

Partial Differential Equations

Nonhomogeneous Neumann problems in Orlicz–Sobolev spaces

Mihai Mihăilescu^{a,b}, Vicențiu Rădulescu^{c,a}

^a University of Craiova, Department of Mathematics, Street A.I. Cuza No. 13, 200585 Craiova, Romania

^b Department of Mathematics, Central European University, 1051 Budapest, Hungary

^c Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

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Abstract

We establish sufficient conditions for the existence of nontrivial solutions for a class of nonlinear Neumann boundary value problems involving nonhomogeneous differential operators. *To cite this article: M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Problèmes de Neumann non homogènes dans les espaces d’Orlicz–Sobolev. On établit des conditions suffisantes pour l’existence des solutions non triviales pour une classe de problèmes aux limites de Neumann avec des opérateurs différentiels non homogènes. *Pour citer cet article : M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Soit $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\operatorname{div}(a(x, |\nabla u(x)|)\nabla u(x)) + a(x, |u(x)|)u(x) = \lambda g(x, u(x)), & \text{pour } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0, & \text{pour } x \in \partial\Omega, \end{cases} \quad (1)$$

où ν est la normale extérieure à $\partial\Omega$. Soit $\phi(x, t) = a(x, |t|)t$ si $t \neq 0$ et $\phi(x, 0) = 0$. On suppose qu’il existe deux constantes ϕ_0 et ϕ^0 telles que

$$1 < \phi_0 \leq \frac{t\phi(x, t)}{\Phi(x, t)} \leq \phi^0 < \infty, \quad \forall x \in \overline{\Omega}, t \geq 0. \quad (2)$$

De plus, on suppose que la fonction Φ satisfait

$$M|t|^{p(x)} \leq \Phi(x, t), \quad \forall x \in \overline{\Omega}, t \geq 0,$$

E-mail addresses: mmihaiiles@yahoo.com (M. Mihăilescu), vicentiu.radulescu@math.cnrs.fr (V. Rădulescu).

URLs: <http://www.inf.ucv.ro/~mihaiilescu> (M. Mihăilescu), <http://www.inf.ucv.ro/~radulescu> (V. Rădulescu).

où $p \in C(\overline{\Omega})$, $p(x) > 1$ pour chaque $x \in \overline{\Omega}$ et $M > 0$ est une constante. D'autre part, on suppose que la fonction g satisfait les conditions

$$|g(x, t)| \leq C_0 |t|^{q(x)-1}, \quad \forall x \in \Omega, t \in \mathbb{R},$$

et

$$C_1 |t|^{q(x)} \leq G(x, t) := \int_0^t g(x, s) ds \leq C_2 |t|^{q(x)}, \quad \forall x \in \Omega, t \in \mathbb{R},$$

où C_0, C_1 et C_2 sont des constantes positives et la fonction $q \in C(\overline{\Omega})$ satisfait $1 < q(x) < \frac{N \min_{\overline{\Omega}} p}{N - \min_{\overline{\Omega}} p}$ pour tout $x \in \overline{\Omega}$.

Le résultat principal de cette Note est le suivant :

Théorème 0.1.

- (i) Si $\min_{\overline{\Omega}} q < \phi_0$ alors il existe $\lambda^* > 0$ tel que pour chaque $\lambda \in (0, \lambda^*)$ le problème (1) admet une solution faible non triviale.
- (ii) Si $\max_{\overline{\Omega}} q < \phi_0$ alors il existe $\lambda^* > 0$ et $\lambda^{**} > 0$ tels que pour chaque $\lambda \in (0, \lambda^*) \cup (\lambda^{**}, \infty)$ le problème (1) admet une solution faible non triviale.

1. The main result

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary. We consider the problem

$$\begin{cases} -\operatorname{div}(a(x, |\nabla u(x)|) \nabla u(x)) + a(x, |u(x)|) u(x) = \lambda g(x, u(x)), & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \quad (3)$$

where ν is the outward unit normal to $\partial\Omega$. In the particular case when $a(x, t) = t^{p(x)-2}$, with p a continuous function on $\overline{\Omega}$, we deal with problems involving variable growth conditions. The study of such problems has been stimulated by recent advances in fluid dynamics (see [3,5,12,13]), image processing (see [1]) and calculus of variations and differential equations with $p(x)$ -growth conditions (see [4–7]).

In this Note we assume that the function $a : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ in (3) is such that the mapping $\phi : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, $\phi(x, t) = a(x, |t|)t$ if $t \neq 0$ and $\phi(x, 0) = 0$ satisfies:

- (ϕ) for all $x \in \Omega$, $\phi(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is an odd, increasing homeomorphism from \mathbb{R} onto \mathbb{R} ; while the function $\Phi : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi(x, t) := \int_0^t \phi(x, s) ds$, for all $x \in \overline{\Omega}$ and all $t \geq 0$ belongs to class Φ (see [9], p. 33), that is, Φ satisfies the following conditions:
- (Φ_1) for all $x \in \Omega$, $\Phi(x, \cdot) : [0, \infty) \rightarrow \mathbb{R}$ is a nondecreasing continuous function, with $\Phi(x, 0) = 0$ and $\Phi(x, t) > 0$ whenever $t > 0$; $\lim_{t \rightarrow \infty} \Phi(x, t) = \infty$;
- (Φ_2) for every $t \geq 0$, $\Phi(\cdot, t) : \Omega \rightarrow \mathbb{R}$ is a measurable function.

Remark 1. Since $\phi(x, \cdot)$ satisfies condition (ϕ) we deduce that $\Phi(x, \cdot)$ is convex and increasing from \mathbb{R}^+ to \mathbb{R}^+ .

For the function Φ introduced above we define the *generalized Orlicz space* $L^\Phi(\Omega)$ as the Banach space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the *Luxemburg norm*

$$|u|_\Phi = \inf \left\{ \mu > 0; \int_{\Omega} \Phi \left(x, \frac{|u(x)|}{\mu} \right) dx \leq 1 \right\},$$

is finite.

In this Note we assume that there exist two positive constants ϕ_0 and ϕ^0 such that

$$1 < \phi_0 \leq \frac{t\phi(x, t)}{\Phi(x, t)} \leq \phi^0 < \infty, \quad \forall x \in \overline{\Omega}, t \geq 0. \quad (4)$$

We point out that in the particular case when $\phi(x, t) = |t|^{p(x)-2}t$ with $p(x) \in C(\overline{\Omega})$ then we denote ϕ^0 by $p^+ := \max_{\overline{\Omega}} p$ and ϕ_0 by $p^- := \max_{\overline{\Omega}} p$.

Furthermore, we assume that Φ satisfies the following condition:

$$\text{for each } x \in \overline{\Omega}, \text{ the function } [0, \infty) \ni t \rightarrow \Phi(x, \sqrt{t}) \text{ is convex.} \tag{5}$$

Remark 2. Relation (5) assures that $L^\Phi(\Omega)$ is an uniformly convex space and thus, a reflexive space.

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x, t) \leq K_1 \cdot \Phi(x, K_2 \cdot t) + h(x), \quad \forall x \in \overline{\Omega}, t \geq 0, \tag{6}$$

where $h \in L^1(\Omega)$, $h(x) \geq 0$ a.e. $x \in \Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [9] we have that there exists the continuous embedding $L^\Phi(\Omega) \subset L^\Psi(\Omega)$.

Next, we build upon $L^\Phi(\Omega)$ the *generalized Orlicz–Sobolev space* $W^{1,\Phi}(\Omega)$ as the space of those weakly differentiable functions in Ω for which the weak derivatives belong to $L^\Phi(\Omega)$. This space endowed with the norm

$$\|u\| = \inf \left\{ \mu > 0; \int_{\Omega} \left[\Phi \left(x, \frac{|u(x)|}{\mu} \right) + \Phi \left(x, \frac{|\nabla u(x)|}{\mu} \right) \right] dx \leq 1 \right\},$$

is a reflexive Banach space. On $W^{1,\Phi}(\Omega)$ the following relations hold true:

$$\int_{\Omega} [\Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)] dx \geq \|u\|^{\phi_0}, \quad \forall u \in W^{1,\Phi}(\Omega) \text{ with } \|u\| > 1; \tag{7}$$

$$\int_{\Omega} [\Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)] dx \geq \|u\|^{\phi^0}, \quad \forall u \in W^{1,\Phi}(\Omega) \text{ with } \|u\| < 1. \tag{8}$$

We refer to Diening [2], Musielak [9], Musielak and Orlicz [10], Nakano [11] for further properties of generalized Orlicz–Sobolev spaces.

In this Note we study problem (3) in the particular case when Φ satisfies

$$M|t|^{p(x)} \leq \Phi(x, t), \quad \forall x \in \overline{\Omega}, t \geq 0, \tag{9}$$

where $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$ and $M > 0$ is a constant.

Remark 3. By relation (9) we deduce that $W^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (6) with $\Psi(x, t) = |t|^{p(x)}$). On the other hand, it is known (see [5]) that $W^{1,p(x)}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x) \in C(\overline{\Omega})$ with $1 < r^- \leq r^+ < \frac{Np^-}{N-p^-}$. Thus, we deduce that $W^{1,\Phi}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x) \in C(\overline{\Omega})$ with $1 < r(x) < \frac{Np^-}{N-p^-}$ for all $x \in \overline{\Omega}$.

On the other hand, we assume that the function g from problem (3) satisfies the hypotheses

$$|g(x, t)| \leq C_0 |t|^{q(x)-1}, \quad \forall x \in \Omega, t \in \mathbb{R} \tag{10}$$

and

$$C_1 |t|^{q(x)} \leq G(x, t) := \int_0^t g(x, s) ds \leq C_2 |t|^{q(x)}, \quad \forall x \in \Omega, t \in \mathbb{R}, \tag{11}$$

where C_0, C_1 and C_2 are positive constants and $q(x) \in C(\overline{\Omega})$ satisfies $1 < q(x) < \frac{Np^-}{N-p^-}$ for all $x \in \overline{\Omega}$.

Example. (a) First, we point out certain examples of functions g and G which satisfy hypotheses (10) and (11).

$$(1) \ g(x, t) = q(x)|t|^{q(x)-2}t \text{ and } G(x, t) = |t|^{q(x)}, \text{ where } q(x) \in C(\overline{\Omega}) \text{ satisfies } 2 \leq q(x) < \frac{Np^-}{N-p^-} \text{ for all } x \in \overline{\Omega};$$

(2) $g(x, t) = q(x)|t|^{q(x)-2}t + (q(x) - 2) \cdot [\log(1 + t^2)]|t|^{q(x)-4}t + \frac{t}{1+t^2}|t|^{q(x)-2}$ and $G(x, t) = |t|^{q(x)} + \log(1 + t^2) \cdot |t|^{q(x)-2}$, where $q(x) \in C(\overline{\Omega})$ satisfies $4 \leq q(x) < \frac{Np^-}{N-p^-}$ for all $x \in \overline{\Omega}$.

(b) Second, we point out certain examples of functions $\phi(x, t)$ and $\Phi(x, t)$ for which the results of this paper can be applied.

(1) $\phi(x, t) = p(x)|t|^{p(x)-2}t$ and $\Phi(x, t) = |t|^{p(x)}$, with $p(x) \in C(\overline{\Omega})$ satisfying $2 \leq p(x) < N$, for all $x \in \overline{\Omega}$.

(2)

$$\phi(x, t) = p(x) \frac{|t|^{p(x)-2}t}{\log(1 + |t|)} \quad \text{and} \quad \Phi(x, t) = \frac{|t|^{p(x)}}{\log(1 + |t|)} + \int_0^{|t|} \frac{s^{p(x)}}{(1 + s)(\log(1 + s))^2} ds$$

with $p(x) \in C(\overline{\Omega})$ satisfying $3 \leq p(x) < N$, for all $x \in \overline{\Omega}$.

(3) $\phi(x, t) = p(x) \log(1 + \alpha + |t|) \cdot |t|^{p(x)-1}t$ and

$$\Phi(x, t) = \log(1 + \alpha + |t|) \cdot |t|^{p(x)} - \int_0^{|t|} \frac{s^{p(x)}}{1 + \alpha + s} dx$$

where $\alpha > 0$ is a constant and $p(x) \in C(\overline{\Omega})$ satisfying $2 \leq p(x) < N$, for all $x \in \overline{\Omega}$.

We say that $u \in W^{1,\Phi}(\Omega)$ is a *weak solution* of problem (3) if

$$\int_{\Omega} a(x, |\nabla u|) \nabla u \nabla v \, dx + \int_{\Omega} a(x, |u|) uv \, dx - \lambda \int_{\Omega} g(x, u) v \, dx = 0,$$

for all $v \in W^{1,\Phi}(\Omega)$.

The main result of this Note is given by the following theorem:

Theorem 1.1. *Assume ϕ and Φ verify conditions (ϕ) , (Φ_1) , (Φ_2) , (4), (5) and (9) and the functions g and G satisfy conditions (10) and (11).*

- (i) *If $q^- < \phi_0$ then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ problem (3) has a nontrivial weak solution.*
- (ii) *If $q^+ < \phi_0$ then there exists $\lambda_* > 0$ and $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$ problem (3) has a nontrivial weak solution.*

Let E denote the generalized Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$.

For each $\lambda > 0$ we define the energy functional $J_\lambda : E \rightarrow \mathbb{R}$ by

$$J_\lambda(u) = \int_{\Omega} [\Phi(x, |\nabla u|) + \Phi(x, |u|)] \, dx - \lambda \int_{\Omega} G(x, u) \, dx, \quad \forall u \in E.$$

Standard arguments imply that J_λ is well-defined on E , $J_\lambda \in C^1(E, \mathbb{R})$ and

$$\langle J'_\lambda(u), v \rangle = \int_{\Omega} a(x, |\nabla u|) \nabla u \cdot \nabla v \, dx + \int_{\Omega} a(x, |u|) uv \, dx - \lambda \int_{\Omega} g(x, u) v \, dx,$$

for all $u, v \in E$. Thus, we remark that the weak solutions of Eq. (3) are exactly the critical points of the energy functional J_λ .

The following auxiliary results will be useful in order to establish the result of Theorem 1.1(i):

Lemma 1.2. *Assume the hypotheses of Theorem 1.1(i) are fulfilled. Then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ there exist $\rho, \alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\| = \rho$.*

Lemma 1.3. *Assume the hypotheses of Theorem 1.1(i) are fulfilled. Then there exists $\theta \in E$ such that $\theta \geq 0$, $\theta \neq 0$ and $J_\lambda(t\theta) < 0$, for $t > 0$ small enough.*

Lemma 1.4. Assume that the sequence $\{u_n\}$ converges weakly to u in E and

$$\limsup_{n \rightarrow \infty} \langle J'_\lambda(u_n), u_n - u \rangle \leq 0.$$

Then $\{u_n\}$ converges strongly to u in E .

Proof of Theorem 1.1(i). Let $\lambda_\star > 0$ be given by Lemma 1.2 and $\lambda \in (0, \lambda_\star)$. By Lemma 1.2 it follows that on the boundary of the ball centered in the origin and of radius ρ in E , denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} J_\lambda > 0$.

On the other hand, by Lemma 1.3, there exists $\theta \in E$ such that $J_\lambda(t\theta) < 0$ for all $t > 0$ small enough. Moreover, relations (8) and (11) and the fact that E is continuously embedded in $L^{q(x)}(\Omega)$ imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \|u\|^{\phi_0} - \lambda C_2 c_1^{q^-} \|u\|^{q^-},$$

where c_1 is a positive constant. It follows that $-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0$.

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\epsilon) < \inf_{B_\rho(0)} J_\lambda + \epsilon \quad \text{and} \quad J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus for small $t > 0$ and any $v \in B_1(0)$ we have

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0 \quad \text{or} \quad \frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \|v\| \geq 0.$$

Letting $t \rightarrow 0$ it follows that $\langle J'_\lambda(u_\epsilon), v \rangle + \epsilon \|v\| > 0$ and we infer that $\|J'_\lambda(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \rightarrow \underline{c} \quad \text{and} \quad J'_\lambda(w_n) \rightarrow 0. \tag{12}$$

It is clear that $\{w_n\}$ is bounded in E . Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E . Using relation (12) we find

$$\lim_{n \rightarrow \infty} \langle J'_\lambda(w_n), w_n - w \rangle = 0.$$

Thus, by Lemma 1.4, we deduce that $\{w_n\}$ converges strongly to w in E . So, by (12), $J_\lambda(w) = \underline{c} < 0$ and $J'_\lambda(w) = 0$. We conclude that w is a nontrivial weak solution for problem (3) for any $\lambda \in (0, \lambda_\star)$. The proof of Theorem 1.1 (i) is complete. \square

Next, we prove Theorem 1.1(ii).

Proof of Theorem 1.1(ii). Since $q^+ < \phi_0$ it follows that $q^- < \phi_0$ and thus, by Theorem 1.1(i) there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ problem (3) has a nontrivial weak solution.

On the other hand, we point out that J_λ is coercive and weakly lower semi-continuous in E , for all $\lambda > 0$. Then Theorem 1.2 in [14] implies that there exists $u_\lambda \in E$ a global minimizer of I_λ and thus a weak solution of problem (3).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and $u_0(x) = t_0$, for all $x \in \Omega$ we have $u_0 \in E$ and

$$J_\lambda(u_0) = \Lambda(u_0) - \lambda \int_\Omega G(x, u_0) \, dx \leq \int_\Omega \Phi(x, t_0) \, dx - \lambda C_1 \int_\Omega |t_0|^{q(x)} \, dx \leq L - \lambda C_1 t_0^{q^+} |\Omega_1|,$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geq \lambda^\star$ and thus u_λ is a nontrivial weak solution of problem (3) for λ large enough. The proof of Theorem 1.1(ii) is complete. \square

We refer to [8] for complete proofs and additional results.

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