Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 143–148
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Ordinary Differential Equations

Non-homogeneous boundary conditions for a fourth-order diffusion equation

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Received 19 September 2007; accepted 6 December 2007

Available online 14 January 2008

Presented by Jean-Michel Bony

Abstract

The existence of classical solutions to a one-dimensional non-linear fourth-order elliptic equation arising in quantum semiconductor modeling is proved for a class of non-homogeneous boundary conditions using degree theory. Furthermore, some non-existence results for other classes of boundary conditions are presented. ***To cite this article: P. Amster et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Conditions aux limites non-homogènes pour une équation diffusive quantique stationnaire en une dimension. L'existence des solutions classiques pour une équation élliptique non-linéaire d'ordre quatre en une dimension, qui apparaît dans la modélisation des semi-conducteurs quantiques, est démontrée pour une classe de conditions aux limites non-homogènes en utilisant la théorie du degré. En plus, des résultats de non-existence pour d'autres classes de conditions aux limites sont établis. ***Pour citer cet article : P. Amster et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Dans la modélisation des semi-conducteurs quantiques ou des fluctuations des interfaces dans des systèmes de spins, on peut dériver l'équation stationnaire (1). Le variable e^v représente une densité des particules (dans le cas des semi-conducteurs) ou une limite de scaling d'une densité de probabilité (dans le cas des systèmes de spins). L'existence des solutions classiques locales en temps de l'équation transiente correspondante avec des conditions aux limites périodiques a été montrée d'abord dans [2]. Puis, l'existence des solutions faibles globales en temps avec des conditions aux limites homogènes de type Dirichlet–Neumann a été démontrée dans [7]. Ce résultat a été généralisé dans [5] pour des conditions aux limites non-homogènes (2). D'autres références peut être trouvé dans [3,6].

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.12.001

Pour des semi-conducteurs quantiques ultra-minces, l'interaction avec l'environnement macroscopique par les contacts $x = 0$ et $x = 1$ peut être très compliquée. C'est pourquoi on propose d'étudier la classe générale des conditions aux limites (3) où $g : \mathbb{R}^8 \to \mathbb{R}^4$ est continue. Comme la non-existence des solutions est possible pour certains choix de g, on a besoin de restrictions sur g. Nous montrons, en utilisant des méthodes du degré topologiques, l'existence des solutions avec des conditions aux limites qui sont, dans une sense à préciser, une perturbation petite de (2).

Plus précisément, on considère une famille de problèmes dépendant d'une paramètre $\lambda \in [0, 1]$. Les solutions de (1), (3) sont obtenues comme les zéros d'une propre fonction F_λ, défini dans (5), pour $\lambda = 1$. Pour appliquer la méthode du degré, il faut définir une homotopie tel que F_0 soit facilement soluble. Alors, soit $\mathcal{V} = C^2(\overline{\Omega}) \times \mathbb{R}^4$ et $\lambda \in [0, 1]$. Nous définissons $T_\lambda : \mathcal{V} \to C^2(\overline{\Omega})$ par $T_\lambda(w, U) = v$ où $v \in C^2(\overline{\Omega})$ est l'unique solution de (4), $B : \mathbb{R}^8 \to \mathbb{R}^4$ est continue et $v_{\partial\Omega} = (v(0), v(1), \dots, v_{xxx}(0), v_{xxx}(1))$. Les conditions aux bords (2) correspondent au choix $B(x_1, \dots, x_8) = (x_1, \dots, x_4)$. Comme T_0 ne dépend pas de w, on peut définir $T_0(U) := T_0(w, U)$. Alors $T_0(U)_{xxxx} = 0$ et $T_0(U)$ est un polynôme d'ordre trois au plus. On suppose :

Hypothèse 1. L'opérateur $T_\lambda : \mathcal{V} \to C^2(\overline{\Omega})$ est bien définie et compacte pour chaque $\lambda \in [0, 1]$.

La fonction $F_\lambda : \mathcal{V} \to C^2(\overline{\Omega}) \times \mathbb{R}^4$ est maintenant définie par (5), où $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ est une fonction continue. On comprend le rôle de F_λ si on considère ces zéros. On a $F_\lambda(v, U) = 0$ si et seulement si (6) est satisfait. En particulier, si $F_1(v, U) = (0, 0)$, les deux prémières équations de (6) impliquent que v est une solution du problème d'origine. En plus, $F_0(v, U) = (0, 0)$ signifie que $\phi(U) = 0$ et v est un polynôme d'ordre trois au plus.

Théorème 0.1. *Suppose l'Hypothèse* 1 *et soient* $\mathcal{U}_1 \subset C^2(\overline{\Omega})$, $\mathcal{U}_2 \subset \mathbb{R}^4$ *des domaines bornées. En plus, on suppose* (7)–(8). *Alors, il existe une solution unique* $v \in \mathcal{U}_1$ *de* (1) *et* $B(v_{\partial\Omega}) = U$.

Dans ce théorème, $\deg_B$ signifie le degré de Brouwer, voir [8] pour des détails. La démonstration du théorème utilise les degrés de Leray–Schauder et Brouwer.

Maintenant, on suppose que B est choisie telle qu'ils existent des estimations a priori.

Hypothèse 2. Pour chaque $K > 0$, il existe une constante $C = C(K)$ tel que pour tout v, U satisfaisant $v = T_\lambda(v, U)$ et $|U| \leqslant K$, on a $\|v\|_{C^2(\overline{\Omega})} \leqslant C$.

Corollaire 0.2. *Suppose les Hypothèses* 1 *et* 2 *avec* $\mathcal{V} = C^2(\overline{\Omega}) \times \mathbb{R}^4$. *Soit* $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ *et* $\psi : \mathbb{R}^8 \to \mathbb{R}^4$ *continues telles que il existe* $R > 0$ *tel que pour tous* $X \in \mathbb{R}^8$ *avec* $|B(X)| > R$, *pour tous* $U \in \mathbb{R}^4$, $|U| > R$, *et pour tous* $\rho > R$, *on a* (9). *Alors, il y a une solution de* (1) *et* (3) *avec* $g(X) = \phi(B(X)) + \psi(X)$.

Ici, $B_\rho(0) = \{r\colon\ |r| < \rho\}$. La condition (e) dans (9) est une condition forte sur les perturbations possibles. C'est naturel, car on a besoin de nouvelles estimations a priori si on modifie les conditions aux limites d'une façon essentielle.

Pour la démonstration du théorème, on montre directement que les conditions du Théorème 0.1 sont satisfaites. Par exemple, si nous choisissons $B(x_1, \dots, x_8) = (x_1, \dots, x_4)$, on peut montrer que les Hypothèses 1 et 2 sont valables. Corollaire 0.2 implique l'existence des solutions de (1) avec des conditions aux limites données par (11), (12) ou par (13) où $J = (e^v v_{xx})_x$ est le courant et $\alpha \in \mathbb{R}$. Ici la condition que $J/\sqrt{1 + J^2}$ est bornée par rapport de J est important.

Naturellement, on ne peut pas choisir des conditions aux limites quelconques et la non-existence des solutions (classiques ou faibles) est possible. On obtient la non-existence des solutions de (1) si, par exemple, les conditions (14), (15) ou (16) sont choisies.

1. Introduction

In the modeling of quantum semiconductor devices and interface fluctuations of spin systems, the following stationary equation has been derived [1,4]:

$$(e^v v_{xx})_{xx} = 0, \quad x \in \Omega = (0, 1). \tag{1}$$

The variable e^v represents a particle density (in case of semiconductor modeling) or a scaling limit of a probability density (in case of spin systems). The existence of local-in-time positive solutions of the corresponding transient equation with periodic boundary conditions was first shown in [2]. The global-in-time existence for non-negative weak solutions with homogeneous Dirichlet–Neumann boundary conditions was then proved in [7]. This result was extended in [5] to the non-homogeneous boundary conditions

$$v(0) = y_0, \quad v(1) = y_1, \quad v_x(0) = z_0, \quad v_x(1) = z_1. \tag{2}$$

Eq. (1) and its transient version were extensively studied in recent years; we refer to [3,6] and references therein.

For ultra-small quantum devices, the interaction with the macroscopic environment through the contacts at $x = 0$ and $x = 1$ can be quite complicated. Therefore, we propose to study the general class of two-point boundary conditions,

$$g(v_{\partial\Omega}) = 0, \quad \text{with } v_{\partial\Omega} = \big(v(0), v(1), v_x(0), v_x(1), v_{xx}(0), v_{xx}(1), v_{xxx}(0), v_{xxx}(1)\big), \tag{3}$$

where $g : \mathbb{R}^8 \to \mathbb{R}^4$ is continuous. Not every choice of g leads to a well-posed problem and thus, some restriction on g is necessary. We show the existence of solutions with boundary conditions which are, in the sense specified below, perturbations of (2) by employing topological degree methods. Moreover, we provide examples of boundary conditions for which (1) does not admit (classical) solutions.

2. An abstract continuation theorem for (1)

We are going to define a family of functions F_λ in (5) below, depending continuously on the homotopy parameter $\lambda \in [0, 1]$. Solutions to (1) and (3) correspond to zeros of F_1, whereas F_0 is such that its Leray–Schauder degree is non-trivial.

Let $\mathcal{V} := C^2(\overline{\Omega}) \times \mathbb{R}^4$ and $\lambda \in [0, 1]$. Define $T_\lambda : \mathcal{V} \to C^2(\overline{\Omega})$ by $T_\lambda(w, U) = v$, where $v \in C^2(\overline{\Omega})$ is the unique solution to

$$\big(e^{\lambda w} v_{xx}\big)_{xx} = 0, \qquad B(v_{\partial\Omega}) = U, \tag{4}$$

where $B : \mathbb{R}^8 \to \mathbb{R}^4$ is continuous and $v_{\partial\Omega}$ is as in (3). For instance, the boundary conditions (2) correspond to the projection $B(x_1, \ldots, x_8) = (x_1, \ldots, x_4)$. Notice that T_0 does not depend on w such that we can write $T_0(U) := T_0(w, U)$.

Assumption 1. The operators $T_\lambda : \mathcal{V} \to C^2(\overline{\Omega})$ are well defined and compact for all $\lambda \in [0, 1]$.

For a given boundary condition $g : \mathbb{R}^8 \to \mathbb{R}^4$ as in (3), and an auxiliary continuous function $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ to be specified later, introduce the functions $F_\lambda : \mathcal{V} \to \mathcal{V}$ by

$$F_\lambda(w, U) = \big(w - T_\lambda(w, U), \lambda g\big(T_\lambda(w, U)_{\partial\Omega}\big) + (1 - \lambda)\phi(U)\big). \tag{5}$$

The role of F_λ is best understood by characterizing its zeros. We have $F_\lambda(v, U) = (0, 0)$ if and only if the following three conditions are satisfied:

$$\big(e^{\lambda v} v_{xx}\big)_{xx} = 0, \quad B(v_{\partial\Omega}) = U, \quad \lambda g(v_{\partial\Omega}) + (1 - \lambda)\phi(U) = 0. \tag{6}$$

In particular, if $F_1(v, U) = (0, 0)$, then v solves the problem (1) and (3). On the other hand, $F_0(v, U) = (0, 0)$ means that v is a polynomial of order $\leqslant 3$ and satisfies the boundary constraint $\phi(B(v_{\partial\Omega})) = 0$.

Theorem 2.1. *Let Assumption* 1 *hold, and let $\mathcal{U}_1 \subset C^2(\overline{\Omega})$, $\mathcal{U}_2 \subset \mathbb{R}^4$ be bounded domains. Assume further, with $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$,*

$$\text{(a)} \quad F_\lambda \neq 0 \text{ on } \partial\mathcal{U}, \quad \text{(b)} \quad \phi \neq 0 \text{ on } \partial\mathcal{U}_2, \quad \text{(c)} \quad \deg_B(\phi, \mathcal{U}_2, 0) \neq 0, \tag{7}$$

$$\text{(d)} \quad rT_0(U) \notin \partial\mathcal{U}_1 \text{ for all } U \in \mathcal{U}_2,\ r \in (0, 1]. \tag{8}$$

Then there exists a solution $v \in \mathcal{U}_1$ to (1)*, satisfying the boundary condition $g(v_{\partial\Omega}) = 0$.*

Here, $\deg_B$ denotes the Brouwer degree, see [8]. Notice that the restrictions on the boundary condition g are implicitly contained in condition (a).

Proof. We have already argued that if $F_1(v,U)=0$ for some $(v,U)\in\mathcal{U}$, then v satisfies (1) and (3). In order to prove the existence of a zero of F_1, we show that its Leray–Schauder degree is non-zero, $\deg_{LS}(F_1,\mathcal{U},0)\neq 0$. By homotopy invariance of the Leray–Schauder degree, and since F_λ does not vanish on $\partial\mathcal{U}$ by (a), it is sufficient to show $\deg_{LS}(F_0,\mathcal{U},0)\neq 0$. To prove the latter, we introduce another homotopy, $H_r:\mathcal{U}\to\mathcal{V}$ for $r\in[0,1]$ by $H_r(v,U)=(v-rT_0(U),\phi(U))$. Notice that $H_1=F_0$ and $H_0(v,U)=(v,U)+(0,\phi(U)-U)$. By definition of the Leray–Schauder degree, $\deg_{LS}(H_0,\mathcal{U},0)=\deg_B(H_0|_{\{0\}\times\mathbb{R}^4},\{0\}\times\mathbb{R}^4,0)=\deg_B(\phi,\mathcal{U}_2,0)$, and the last degree does not vanish by (c). By homotopy invariance, H_1 has non-zero degree if H_r does not vanish on $\partial\mathcal{U}$. Assume, by contradiction, that $H_r(v,U)=(0,0)$ for some $(v,U)\in\partial\mathcal{U}$. Then $\phi(U)=0$, and hence $U\in\mathcal{U}_2$, $v\in\partial\mathcal{U}_1$ employing (b). However, $v=rT_0(U)\in\partial\mathcal{U}_1$ contradicts (d). □

The most relevant case for applications is when the hypotheses of Theorem 2.1 follow from a priori estimates on the solution v to (1) with the unperturbed boundary condition $B(v_{\partial\Omega})=U$.

Assumption 2. For each $K>0$, there exists a constant $C=C(K)$, such that for all v, U satisfying $v=T_\lambda(v,U)$ and $|U|\leqslant K$, it follows $\|v\|_{C^2(\overline{\Omega})}\leqslant C$.

Corollary 2.2. *Let Assumptions* 1 *and* 2 *hold. Furthermore, let $\phi:\mathbb{R}^4\to\mathbb{R}^4$ and $\psi:\mathbb{R}^8\to\mathbb{R}^4$ be continuous functions such that there exists an $R>0$ such that for all $X\in\mathbb{R}^8$ with $|B(X)|>R$, for all $U\in\mathbb{R}^4$ with $|U|>R$, and for all $\rho>R$, the following conditions are satisfied*:

$$\text{(e)}\quad \bigl|\psi(X)\bigr|<\bigl|\phi\bigl(B(X)\bigr)\bigr|,\qquad \text{(f)}\quad \phi(U)\neq 0,\qquad \text{(g)}\quad \deg_B\bigl(\phi,B_\rho(0),0\bigr)\neq 0. \tag{9}$$

Then there exists at least one solution to (1) *and* (3) *with $g(X)=\phi(B(X))+\psi(X)$.*

Here, $B_\rho(0)=\{r\colon |r|<\rho\}$. Condition (e) is quite restrictive on the allowed perturbations. This is natural since any essential change of the boundary conditions will require genuinely new a priori estimates.

Proof. First we prove the existence of a number $R_0>0$ such that $F_\lambda(v,U)=0$ implies $\|v\|_{C^2(\overline{\Omega})}<R_0$ and $|U|<R_0$. Suppose, on the contrary, that there exist sequences $(\lambda_n)\subset[0,1]$, $(U_n)\subset\mathbb{R}^4$, and $(v_n)\subset C^2(\overline{\Omega})$ such that $F_{\lambda_n}(v_n,U_n)=0$ but $\|v_n\|_{C^2(\overline{\Omega})}+|U_n|\to\infty$ as $n\to\infty$. If $|U_n|\leqslant K$ for some $K>0$ and all $n\in\mathbb{N}$, we deduce from Assumption 2 that $\|v_n\|_{C^2(\overline{\Omega})}\leqslant C(K)$ and so, $\|v_n\|_{C^2(\overline{\Omega})}+|U_n|$ would be bounded; contradiction. Therefore, $|U_n|\to\infty$. Moreover, recalling the last equation in (6), $F_{\lambda_n}(v_n,U_n)=0$ implies $\lambda_n g((v_n)_{\partial\Omega})+(1-\lambda_n)\phi(U_n)=0$. Combining this with the second equation in (6), we arrive at $\phi(B((v_n)_{\partial\Omega}))+\lambda_n\psi((v_n)_{\partial\Omega})=0$. Since $\lambda\in[0,1]$ and $|U_n|\to\infty$, condition (e) implies that $\phi(B((v_n)_{\partial\Omega}))=0$ for n large enough. Hence also $\phi(U_n)=0$. However, this contradicts, for sufficiently large $n\in\mathbb{N}$, condition (f). Take $\mathcal{U}_1=B_{R_1}(0)$ and $\mathcal{U}_2=B_{R_2}(0)$ with $R_2>\max\{R_0,R\}$ and $R_1>\max\{R_0,\sup_{|U|\leqslant R_2}\|T_0(U)\|_{C^2(\overline{\Omega})}\}$. From the previous computations, it follows that condition (a) of Theorem 2.1 holds. Moreover, (f) and (g) imply the validity of (b) and (c). Finally, the choice of R_1 shows that $rT_0(U)\notin\partial\mathcal{U}_1$ for any $U\in\mathcal{U}_2$, $r\in(0,1]$. Thus, all conditions of Theorem 2.1 are satisfied. □

3. Perturbations of the Dirichlet–Neumann conditions

In order to specialize our abstract result, we choose $B(x_1,\dots,x_8)=(x_1,\dots,x_4)$ which corresponds to the Dirichlet–Neumann boundary conditions (2).

Lemma 3.1. *With the above choice of B, Assumptions* 1 *and* 2 *hold.*

Proof. First we verify Assumption 1. Let $v=T_\lambda(w,y_0,y_1,z_0,z_1)$. Then v solves (4). Thus, $v_{xx}=(ax+b)e^{-\lambda w}$ for some constants $a,b\in\mathbb{R}$. Conversely, given $a,b\in\mathbb{R}$ and defining v as the solution of this differential equation with

$v(0) = y_0$ and $v_x(0) = z_0$, then v is a solution to (4) if and only if $v(1) = y_1$ and $v_x(1) = z_1$. In other words, (4) has a unique solution v if and only if the linear system

$$\begin{pmatrix} g(1) & h(1) \\ G(1) & H(1) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} z_1 - z_0 \\ y_1 - y_0 - z_0 \end{pmatrix} \tag{10}$$

has a unique solution, where

$$g(x) = \int_0^x \xi e^{-\lambda w(\xi)} \, d\xi, \quad h(x) = \int_0^x e^{-\lambda w(\xi)} \, d\xi, \quad G(x) = \int_0^x g(\xi) \, d\xi, \quad H(x) = \int_0^x h(\xi) \, d\xi.$$

A simple computation shows that g/h is strictly increasing on $\overline{\Omega}$. Thus,

$$G(1) = \int_0^1 g(\xi) \, d\xi < \frac{g(1)}{h(1)} \int_0^1 h(\xi) \, d\xi = \frac{g(1)}{h(1)} H(1),$$

which implies that the determinant of the coefficient matrix in (10) is positive, and T_λ is well-defined.

Obviously, the solution (a, b) to (10) depends continuously on the data $y = (y_0, y_1)$, $z = (z_0, z_1)$ and w. Hence, differentiation of $v_{xx} = (ax + b)e^{-\lambda w}$ yields a bound for $|v_{xxx}|$ depending continuously on $|y|$, $|z|$ and $\|w\|_{C^1(\overline{\Omega})}$. Finally, the compactness of the embedding $C^3(\overline{\Omega}) \hookrightarrow C^2(\overline{\Omega})$ guarantees the compactness of T_λ. Thus, Assumption 1 holds.

In principle, Assumption 2 can be deduced from the results of [5]. However, we prefer to give a direct proof, which is short and adapted to the current framework. Let $v \in C^2(\overline{\Omega})$ satisfy $v = T_\lambda(v, y, z)$. Then there exist constants a, $b \in \mathbb{R}$ such that $v_{xx} = e^{-\lambda v}(ax + b)$. Thus, v_{xx} can change its sign at most once. We wish to estimate the minimal value $v_- = \inf\{v(x): x \in \Omega\}$. If v_- is attained at $\partial\Omega$ then $v_- \geqslant -(|y_0| + |y_1|)$. On the other hand, let $v_- = v(x_-)$ for some $x_- \in \Omega$. Since v_{xx} changes sign at most once, x_- is uniquely determined. We can assume without loss of generality that v is convex on $(0, x_-)$ (otherwise consider $\tilde{v}: x \mapsto v(1 - x)$ instead of v). Convexity implies that $v_x(x) \geqslant v_x(0) = z_0$ for all $x \in (0, x_-)$. Hence, $v_- \geqslant -(|y_0| + |z_0|)$. In a similar way, we can prove a bound on the maximal value v_+. Thus, $\sup_\Omega |v| \leqslant K := |y_0| + |y_1| + |z_0| + |z_1|$.

Next, we estimate $\ell(x) = ax + b$. This function achieves its maximum and minimum at the boundary; assume without loss of generality that $|\ell(0)| = L := \sup_\Omega |\ell|$. Then $|\ell(x)| \geqslant L(1 - 2x)$ for $x \in (0, 1/2)$, and so, by Taylor's expansion,

$$3K \geqslant \left| v\left(\frac{1}{2}\right) - v(0) - \frac{1}{2} v_x(0) \right| = \int_0^{1/2} \int_0^x \left|\ell(y)\right| e^{-\lambda v(y)} \, dy \, dx \geqslant L e^{-\lambda K} \int_0^{1/2} \int_0^x (1 - 2y) \, dy \, dx = \frac{L}{12} e^{-\lambda K}.$$

We conclude that $L \leqslant 36K e^{\lambda K}$. Thus, $\sup_\Omega |v_{xx}| \leqslant \sup_\Omega |ax + b| \sup_\Omega e^{-\lambda v} \leqslant 36K e^{2\lambda K}$. Since the values of v and v_x are prescribed on $\partial\Omega$, the desired C^2 estimate follows and Assumption 2 is proved. □

Example 1. Corollary 2.2 allows one to conclude the existence of solutions to (1) with boundary conditions

$$a_{i1} v(0) + a_{i2} v(1) + a_{i3} v_x(0) + a_{i4} v_x(1) = \psi_i(v_{\partial\Omega}),$$

where $A = (a_{ij})_{ij} \in \mathbb{R}^{4\times 4}$ is an arbitrary regular matrix, and each $\psi_i : \mathbb{R}^8 \to \mathbb{R}$, $i = 1, \ldots, 4$, is a bounded continuous function. To give a specific example, let $\epsilon > 0$ and define

$$\phi(x_1, \ldots, x_4) = \epsilon(x_1, \ldots, x_4), \qquad \psi_i(x_1, \ldots, x_8) = h_\epsilon(x_{i+5}) - z_i,$$

corresponding to

$$h_\epsilon\big(v_{xx}(0)\big) = z_1 - \epsilon v(0), \qquad h_\epsilon\big(v_{xxx}(0)\big) = z_3 - \epsilon v_x(0), \tag{11}$$

$$h_\epsilon\big(v_{xx}(1)\big) = z_2 - \epsilon v(1), \qquad h_\epsilon\big(v_{xxx}(1)\big) = z_4 - \epsilon v_x(1), \tag{12}$$

with arbitrary numbers $z_1, \ldots, z_4 \in \mathbb{R}$ and $h_\epsilon(y) = \epsilon^{-1} \arctan(\epsilon y)$. We obtain the existence of solutions to (1), (11) and (12) for any $\epsilon > 0$. However, we cannot conclude the solvability of (1) under the boundary conditions

$$v_{xx}(0) = z_1, \quad v_{xx}(1) = z_2, \quad v_{xxx}(0) = z_3, \quad v_{xxx}(1) = z_4,$$

which are obtained from (11) in the limit $\epsilon \searrow 0$, without further a priori estimates.

Another admissible set of boundary conditions are the higher-order Robin-type conditions

$$v + \alpha \frac{J}{\sqrt{1+J^2}} = v_0 \quad \text{on } \partial\Omega, \qquad v_x(0) = z_0, \quad v_x(1) = z_1, \tag{13}$$

where $J = (e^v v_{xx})_x$ is the flux and $\alpha \in \mathbb{R}$. Notice that $J/\sqrt{1+J^2}$ is bounded on $\mathbb{R}$.

Finally, we provide examples of boundary conditions for which (1) does not admit a (classical) solution:

$$\text{(i)} \quad J(0) = J_0, \quad v(1) = y_1, \quad v_{xx}(0) = 0, \quad v_{xx}(1) = \alpha_0, \quad \text{where } \alpha_0 \neq J_0 e^{-y_1}; \tag{14}$$

$$\text{(ii)} \quad v_x(0) = z_0, \quad v_x(1) = z_1, \quad v_{xx}(0) = \alpha_0, \quad v_{xx}(1) = \alpha_1, \quad \text{where } z_0 > z_1,\ \alpha_0, \alpha_1 > 0; \tag{15}$$

$$\text{(iii)} \quad v(0) = v(1), \quad v_x(0) = z_0, \quad v_x(1) = z_1, \quad v_{xx}(0) = v_{xx}(1), \quad \text{where } |z_0| \neq |z_1|. \tag{16}$$

This can be seen as follows. Suppose that there exists a solution to (1) with (i). Then J is constant in Ω and $e^v v_{xx} = J_0 x + b$. The boundary condition $v_{xx}(0) = 0$ implies that $b = 0$. Thus $v_{xx}(1) = e^{-y_1} J_0$, contradiction. Let v solve (1) with (ii). By the mean-value theorem, there exists $\xi \in (0,1)$ such that $v_{xx}(\xi) = v_x(1) - v_x(0) < 0$. But v_{xx} can change its sign only once; contradiction. Finally, let v solve (1) with (iii). Then $v(0) = v(1)$ and $v_{xx}(0) = v_{xx}(1)$ imply $v_{xx} = be^{-v}$; an integration yields the contradiction

$$v_x(1)^2 - v_x(0)^2 = 2\int_0^1 v_{xx} v_x \, \mathrm{d}x = 2b \int_0^1 e^{-v} v_x \, \mathrm{d}x = 2b\left(e^{-v(0)} - e^{-v(1)}\right) = 0.$$

Acknowledgements

The authors acknowledge partial support from the DAAD-Secyt Project. A.J. and D.M. have been supported by the DFG, grants JU 359/5 and JU 359/7, by the FWF and the ESF Program GLOBAL.

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