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Ordinary Differential Equations

A mathematical framework for a crowd motion model

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Abstract

In a previous paper, we proposed a model for crowd motion, together with a numerical algorithm, especially designed to handle highly packed situations. This model rests on two principles: We first define a spontaneous velocity which corresponds to the velocity each individual would like to have in the absence of other people; The actual velocity is then computed as the projection of the spontaneous velocity onto the set of admissible velocities (i.e. velocities which do not violate the non-overlapping constraint). We describe here the underlying mathematical framework, and we explain how recent results by J.F. Edmond and L. Thibault on the sweeping process in the prox-regular case can be adapted to handle this situation, in terms of well-posedness as well as convergence of the numerical algorithm. ***To cite this article: B. Maury, J. Venel, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Cadre mathématique pour un modèle de mouvement de foules. Dans un précédent papier, nous avons proposé un modèle de mouvements de foule ainsi qu'un algorithme numérique, ayant pour objectif de gérer des configurations très denses. Ce modèle repose sur deux principes. D'une part, on définit une vitesse souhaitée qui correspond à la vitesse que les individus aimeraient avoir en l'absence des autres. D'autre part, la vitesse réelle est calculée comme la projection de la vitesse souhaitée sur l'ensemble des vitesses admissibles (vitesses qui respectent la contrainte de non-chevauchement). Nous décrivons ici le cadre mathématique sous-jacent et expliquons comment certains résultats récents de J.F. Edmond et L. Thibault sur les processus de rafle dans le cadre prox-régulier peuvent être adaptés pour démontrer le caractère bien posé du problème et la convergence du schéma numérique associé. ***Pour citer cet article : B. Maury, J. Venel, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Nous considérons n personnes, identifiées à des disques rigides de rayon r et de centre $\mathbf{q}_i$. L'espace des configurations admissibles (sans recouvrement) est défini par

$$Q = \left\{\mathbf{q} \in \mathbb{R}^{2n},\ D_{ij}(\mathbf{q}) = |\mathbf{q}_i - \mathbf{q}_j| - 2r \geqslant 0\ \forall i \neq j\right\}.$$

Nous nous plaçons dans le cas où les personnes ont toutes le même comportement, de telle sorte que le champ de vitesse souhaitée est de la forme

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$$\mathbf{U}(\mathbf{q}) = \big(\mathbf{U}_0(\mathbf{q}_1), \ldots, \mathbf{U}_0(\mathbf{q}_n)\big) \in \mathbb{R}^{2n},$$

où $\mathbf{U}_0(\cdot) \in \mathbb{R}^2$ est un champ de vitesse donné. L'ensemble des vitesses admissibles s'écrit

$$\mathcal{C}_\mathbf{q} = \big\{\mathbf{v} \in \mathbb{R}^{2n},\ \forall i < j\ D_{ij}(\mathbf{q}) = 0 \Rightarrow \mathbf{G}_{ij}(\mathbf{q}) \cdot \mathbf{v} \geqslant 0\big\},$$
$$\text{avec } \mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q}) = \big(0, \ldots, 0, -\mathbf{e}_{ij}(\mathbf{q}), 0, \ldots, 0, \mathbf{e}_{ij}(\mathbf{q}), 0, \ldots, 0\big) \in \mathbb{R}^{2n}.$$

La vitesse effective de la collection d'individus est alors définie par

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathrm{P}_{\mathcal{C}_\mathbf{q}} \mathbf{U}(\mathbf{q}),$$

où $\mathrm{P}_{\mathcal{C}_\mathbf{q}}$ représente la projection (ℓ^2) sur le cône convexe fermé $\mathcal{C}_\mathbf{q}$.

On définit $\mathcal{N}_\mathbf{q}$, cône normal sortant à l'ensemble des configurations admissibles Q, comme le cône polaire de $\mathcal{C}_\mathbf{q}$:

$$\mathcal{N}_\mathbf{q} = \mathcal{C}_\mathbf{q}^o = \{\mathbf{w},\ \mathbf{w} \cdot \mathbf{v} \leqslant 0\ \forall \mathbf{v} \in \mathcal{C}_\mathbf{q}\},$$

qui s'exprime

$$\mathcal{N}_\mathbf{q} = \Big\{-\sum \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}),\ \lambda_{ij} \geqslant 0,\ D_{ij}(\mathbf{q}) > 0 \Longrightarrow \lambda_{ij} = 0\Big\},$$

ce qui permet d'obtenir une nouvelle formulation du problème sous la forme d'une inclusion différentielle

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} + \mathcal{N}_\mathbf{q} \ni \mathbf{U}(\mathbf{q}). \tag{1}$$

Comme Q n'est pas convexe, l'opérateur multivalué $\mathbf{q} \mapsto \mathcal{N}_\mathbf{q}$ n'est pas monotone. Néanmoins, le caractère prox-régulier de Q (voir [4]) permet d'assurer le caractère bien posé du problème, grâce à des résultats récents de [3,4] sur le processus de rafle (voir Théorème 2.4 ci-après).

Le schéma numérique proposé dans [5] est basé sur une approximation de l'ensemble des configurations admissibles Q (non convexe) par l'ensemble

$$\tilde{Q}(\mathbf{q}) = \big\{\tilde{\mathbf{q}},\ D_{ij}(\mathbf{q}) + \mathbf{G}_{ij}(\mathbf{q}) \cdot (\tilde{\mathbf{q}} - \mathbf{q}) \geqslant 0\big\}$$

qui peut être vu comme une approximation intérieure convexe de Q relativement à $\mathbf{q}$. Le schéma peut alors s'écrire

$$\frac{\mathbf{q}^{k+1} - \mathbf{q}^k}{h} + \partial I_{\tilde{Q}(\mathbf{q}^k)}\big(\mathbf{q}^{k+1}\big) \ni \mathbf{U}\big(\mathbf{q}^k\big). \tag{2}$$

Ici encore, au prix de certaines difficultés techniques liées au remplacement de Q par $\tilde{Q}$, une démarche voisine de celle menée dans [4] permet de montrer la convergence du schéma numérique (Théorème 3.2).

1. Modelling

We consider n persons identified to rigid disks. For the sake of simplicity, the disks are supposed here to have the same radius r. The centre of the ith disk is denoted by $\mathbf{q}_i$ (see Fig. 1). Since overlapping is forbidden, the vector of positions $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n}$ must belong to the set of feasible configurations:

$$Q = \big\{\mathbf{q} \in \mathbb{R}^{2n},\ D_{ij}(\mathbf{q}) = |\mathbf{q}_i - \mathbf{q}_j| - 2r \geqslant 0\ \forall i \neq j\big\}.$$

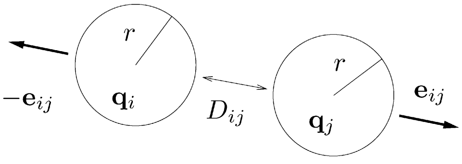

Fig. 1. Notations.

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We consider that the individuals have the same behavior, so that the spontaneous velocity of the n persons can be written:

$$\mathbf{U}(\mathbf{q}) = \big(\mathbf{U}_0(\mathbf{q}_1), \ldots, \mathbf{U}_0(\mathbf{q}_n)\big) \in \mathbb{R}^{2n},$$

where $\mathbf{U}_0(\cdot) \in \mathbb{R}^2$ is a global spontaneous velocity field. The case where an individual may have different spontaneous velocities is handled in [8]. Owing to the non-overlapping constraint, the set of feasible velocities is defined as:

$$\mathcal{C}_\mathbf{q} = \big\{\mathbf{v} \in \mathbb{R}^{2n},\ \forall i < j\ D_{ij}(\mathbf{q}) = 0\ \Rightarrow\ \mathbf{G}_{ij}(\mathbf{q}) \cdot \mathbf{v} \geqslant 0\big\},$$
$$\text{with } \mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q}) = \big(0, \ldots, 0, -\mathbf{e}_{ij}(\mathbf{q}), 0, \ldots, 0, \mathbf{e}_{ij}(\mathbf{q}), 0, \ldots, 0\big) \in \mathbb{R}^{2n}.$$

The actual velocity field is defined as the feasible field which is the closest to $\mathbf{U}$ in the least square sense, which writes

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathrm{P}_{\mathcal{C}_\mathbf{q}} \mathbf{U}(\mathbf{q}),$$

where $\mathrm{P}_{\mathcal{C}_\mathbf{q}}$ denotes the Euclidean (ℓ^2) projection onto the closed convex cone $\mathcal{C}_\mathbf{q}$.

2. Mathematical framework

Despite its formal simplicity, this model does not fit directly into a standard framework. Let us reformulate the problem by introducing $\mathcal{N}_\mathbf{q}$, the outward normal cone to the set of feasible configurations Q, which is defined as the polar cone of $\mathcal{C}_\mathbf{q}$

$$\mathcal{N}_\mathbf{q} = \mathcal{C}_\mathbf{q}^o = \{\mathbf{w},\ \mathbf{w} \cdot \mathbf{v} \leqslant 0\ \forall \mathbf{v} \in \mathcal{C}_\mathbf{q}\}.$$

Thanks to Farkas' Lemma, this cone can be expressed

$$\mathcal{N}_\mathbf{q} = \Big\{-\sum \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}),\ \lambda_{ij} \geqslant 0,\ D_{ij}(\mathbf{q}) > 0 \Longrightarrow \lambda_{ij} = 0\Big\}.$$

Now using the classical orthogonal decomposition of a Hilbert space as the sum of mutually polar cone (see [7]), we obtain

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathrm{P}_{\mathcal{C}_\mathbf{q}} \mathbf{U}(\mathbf{q}) = \mathbf{U}(\mathbf{q}) - \mathrm{P}_{\mathcal{N}_\mathbf{q}} \mathbf{U}(\mathbf{q}).$$

As a consequence,

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} + \mathcal{N}_\mathbf{q} \ni \mathbf{U}(\mathbf{q}). \tag{3}$$

The problem reads as a first order differential inclusion, which has motivated a huge amount of papers in the last decades. Let us first study a special situation where standard theory can be applied. Consider n individuals in a corridor. In that case, as people cannot leap across each other, it is natural to restrict the set of feasible configurations to one of its connected components:

$$Q = \big\{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n,\ q_{i+1} - q_i \geqslant 2r\big\}.$$

In this very situation, as Q is closed and convex, the multivalued operator $\mathbf{q} \mapsto \mathcal{N}_\mathbf{q}$ identifies to the subdifferential of the indicator function of Q:

$$\partial I_Q(\mathbf{q}) = \big\{\mathbf{v}, I_Q(\mathbf{q}) + (\mathbf{v}, \mathbf{h}) \leqslant I_Q(\mathbf{q} + \mathbf{h})\ \forall \mathbf{h}\big\}, \quad I_Q(\mathbf{q}) = \left| \begin{array}{ll} 0 & \text{if } \mathbf{q} \in Q, \\ +\infty & \text{if } \mathbf{q} \notin Q \end{array} \right.$$

therefore $\mathbf{q} \mapsto \mathcal{N}_\mathbf{q}$ is maximal monotone. In that case, as soon as the spontaneous velocity is regular (say Lipschitz), standard theory (see e.g. Brezis [1]) ensures well-posedness. In the general case, Q is not convex and $\mathbf{q} \mapsto \mathcal{N}_\mathbf{q}$ is not monotone. Yet, Q is uniformly prox-regular, which is the suitable property to ensure well-posedness. Let us give some definitions to specify this point.

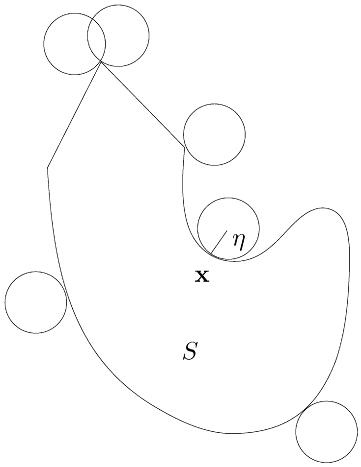

Fig. 2. η-prox-regular set.

Fig. 2. Ensemble η-prox-régulier.

Definition 2.1. Let S be a closed subset of a Hilbert space H.

We define the proximal normal cone to S at $\mathbf{x}$ by:

$$\mathrm{N}(S,\mathbf{x}) = \left\{\mathbf{v} \in H,\ \exists\alpha > 0,\ \mathbf{x} \in \mathrm{P}_S(\mathbf{x}+\alpha\mathbf{v})\right\},$$

where

$$\mathrm{P}_S(\mathbf{y}) = \left\{\mathbf{z} \in S,\ d(\mathbf{y},S) = |\mathbf{y}-\mathbf{z}|\right\}.$$

S is uniformly prox-regular with constant $\eta > 0$ if for all $\mathbf{x} \in \partial S$ and $\mathbf{v} \in \mathrm{N}(S,\mathbf{x})$, $|\mathbf{v}| = 1$ we have:

$$B(\mathbf{x}+\eta\mathbf{v},\eta) \cap S = \emptyset.$$

This definition ensures that the projection onto such a set is well-defined in its neighborhood (see Fig. 2).

The following propositions show that the differential inclusion (3) fits into the sweeping process framework (for details about sweeping process see [4]):

Proposition 2.2. *We consider the following set*

$$Q_{ij} = \left\{\mathbf{q} \in \mathbb{R}^{2n},\ D_{ij}(\mathbf{q}) \geqslant 0\right\}.$$

Q_{ij} *is uniformly prox-regular with constant* $r\sqrt{2}$ *and for all* $\mathbf{q} \in Q_{ij}$,

$$\mathrm{N}(Q_{ij},\mathbf{q}) = -\mathbb{R}^+\mathbf{G}_{ij}(\mathbf{q}).$$

Sketch of the proof. As the boundary of Q_{ij} is smooth, the cone $\mathrm{N}(Q_{ij},\mathbf{q})$ is generated by the outward normal vector to Q_{ij} at point $\mathbf{q} \in \partial Q_{ij}$ which is the renormalized vector $-\mathbf{G}_{ij}(\mathbf{q})$. Consequently,

$$\mathrm{N}(Q_{ij},\mathbf{q}) = -\mathbb{R}^+\mathbf{G}_{ij}(\mathbf{q}).$$

Moreover, the constant of prox-regularity corresponds to the smallest radius of curvature i.e. the largest eigenvalue of Weingarten operator. After calculation, we show that Q_{ij} is uniformly prox-regular with constant $r\sqrt{2}$. □

The set of feasible configurations Q, that is, the intersection of all sets Q_{ij}, has a similar property.

Proposition 2.3. Q *is uniformly prox-regular and for all* $\mathbf{q} \in Q$,

$$\mathcal{N}_\mathbf{q} = \mathrm{N}(Q,\mathbf{q}).$$

Here the results of differential geometry cannot be applied. The proof of this proposition is quite technical and will be postponed to a forthcoming paper. However we can give an upper bound of η, which is obtained by considering a particular configuration (n disks in a row):

$$\eta \leqslant r\sqrt{\frac{12}{n(n-1)(n+1)}}.$$

It is to be noticed that parameter η (which quantifies the prox-regularity of Q) goes to 0 as n goes to infinity or as r goes to zero, which indicates an asymptotic degenerate behavior as the population size increases.

The prox-regular character of Q makes it possible to use the results in [3,4] to establish well-posedness:

Theorem 2.4. *Assume that U is Lipschitz. Then, for all $\mathbf{q}_0 \in Q$, the following problem*

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} + \mathcal{N}_{\mathbf{q}} \ni \mathbf{U}(\mathbf{q}), \\ \mathbf{q}(0) = \mathbf{q}_0, \end{cases}$$

has one and only one absolutely continuous solution $\mathbf{q}(\cdot)$.

3. Numerical scheme

We present in this section a numerical scheme to approximate the solution to (3) and we indicate how convergence can be proved. For numerical simulations using this algorithm, we refer the reader to [5,6,8].

The numerical scheme we propose is based on a first order expansion of the constraints expressed in terms of velocities. The time interval is denoted by $[0, T]$. Let $h = T/p$ be the time step and $t^k = kh$ the computational times. We denote by $\mathbf{q}^k$ the approximation of $\mathbf{q}(t^k)$. The next configuration is obtained as

$$\mathbf{q}^{k+1} = \mathbf{q}^k + h\,\mathbf{u}^{k+1},$$

where

$$\mathbf{u}^{k+1} = \mathrm{P}_{\mathcal{C}^h_{\mathbf{q}^k}}\big(\mathbf{U}\big(\mathbf{q}^k\big)\big) \quad \text{with}$$

$$\mathcal{C}^h_{\mathbf{q}} = \big\{\mathbf{v},\ D_{ij}(\mathbf{q}) + h\mathbf{G}_{ij}(\mathbf{q})\cdot\mathbf{v} \geqslant 0\big\}.$$

The scheme can be also interpreted in the following way. Let us introduce the set

$$\tilde{Q}(\mathbf{q}) = \big\{\tilde{\mathbf{q}},\ D_{ij}(\mathbf{q}) + \mathbf{G}_{ij}(\mathbf{q})\cdot(\tilde{\mathbf{q}} - \mathbf{q}) \geqslant 0\big\},$$

which can be seen as an inner convex approximation of Q with respect to $\mathbf{q}$. Note that $\tilde{Q}(\mathbf{q})$ is defined in such a way that Q is the union of all sets $\tilde{Q}(\mathbf{q})$, $\mathbf{q} \in Q$. It is straightforward to check that

$$\frac{\mathbf{q}^{k+1} - \mathbf{q}^k}{h} + \partial I_{\tilde{Q}(\mathbf{q}^k)}\big(\mathbf{q}^{k+1}\big) \ni \mathbf{U}\big(\mathbf{q}^k\big), \tag{4}$$

so that the scheme can be seen as a semi-implicit discretization of (3), where $\partial I_{\tilde{Q}(\mathbf{q}^k)}(\mathbf{q}^{k+1})$ approximates $\mathcal{N}_{\mathbf{q}^k}$. More precisely, $\tilde{Q}(\mathbf{q})$ is an accurate local approximation of Q at point $\mathbf{q}$ in the following sense:

Lemma 3.1. *For all $\mathbf{q} \in Q$,*

$$\mathrm{N}(Q, \mathbf{q}) = \mathcal{N}_{\mathbf{q}} = \partial I_{\tilde{Q}(\mathbf{q})}(\mathbf{q}) = \mathrm{N}\big(\tilde{Q}(\mathbf{q}), \mathbf{q}\big).$$

It can be shown that the solution to this numerical scheme converges to the exact solution:

Theorem 3.2. *Let us denote by $\mathbf{q}_h$ the continuous, piecewise linear function associated to the numerical scheme. Then $\mathbf{q}_h$ goes to $\mathbf{q}$ uniformly in $[0, T]$, where $t \mapsto \mathbf{q}(t)$ is the exact solution of (3).*

Sketch of the proof. First, a convergent subsequence can be easily extracted. Next, it remains to prove that the limit function satisfies (3). To do so, the idea is to go to the limit in (4) using tools developed in [4]. However, in our case, a new difficulty raises from the fact that sets $\tilde{Q}(\mathbf{q}^k)$ do not depend in a regular way on time. Thus, we are lead to study the convergence of the associated projection operators. The following lemma shows that these operators are locally regular in a certain way, which is sufficient to complete the proof of the theorem. □

Lemma 3.3. *Let* $\mathbf{q} \in Q$ *and a sequence* $(\mathbf{q}_k)$ *converging to* $\mathbf{q}$. *For all* $\tilde{\mathbf{q}} \in \mathbb{R}^{2n}$, *we denote* $\tilde{\mathbf{p}}$ *(respectively* $\widetilde{\mathbf{p}_k}$*) the projection of* $\tilde{\mathbf{q}}$ *onto* $\tilde{Q}(\mathbf{q})$ *(respectively onto* $\tilde{Q}(\mathbf{q}_k)$*). Then there exists* $\nu > 0$ *so that for all* $\tilde{\mathbf{q}} \in B(\mathbf{q}, \nu)$, *the sequence* $(\widetilde{\mathbf{p}_k})$ *converges to* $\tilde{\mathbf{p}}$.

Sketch of the proof. We reformulate the problem of projection as follows:

$$\tilde{\mathbf{p}} = \mathrm{P}_{\tilde{Q}(\mathbf{q})}(\tilde{\mathbf{q}})$$

which is equivalent to

$$\tilde{\mathbf{p}} = \operatorname*{argmin}_{\mathbf{p} \in \tilde{Q}(\mathbf{q})} \frac{1}{2}|\mathbf{p} - \tilde{\mathbf{q}}|^2. \tag{5}$$

We introduce the associated Lagrangian:

$$L(\mathbf{p}, \boldsymbol{\mu}) = \frac{1}{2}|\mathbf{p} - \tilde{\mathbf{q}}|^2 - \sum_{1 \leqslant i < j \leqslant n} \mu_{ij}\big(D_{ij}(\mathbf{q}) + \mathbf{G}_{ij}(\mathbf{q}) \cdot (\mathbf{p} - \mathbf{q})\big).$$

The existence of a saddle-point $(\tilde{\mathbf{p}}, \boldsymbol{\lambda})$ for this problem is well-known (see e.g. [2]) and it is characterized by the following system:

$$\begin{cases} \tilde{\mathbf{p}} = \tilde{\mathbf{q}} + \sum_{i<j} \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}), \\ \forall i < j,\ D_{ij}(\mathbf{q}) + \mathbf{G}_{ij}(\mathbf{q}) \cdot (\tilde{\mathbf{p}} - \mathbf{q}) \geqslant 0, \\ \sum_{i<j} \lambda_{ij}\ \big(D_{ij}(\mathbf{q}) + \mathbf{G}_{ij}(\mathbf{q}) \cdot (\tilde{\mathbf{p}} - \mathbf{q})\big) = 0. \end{cases}$$

We obtain similar systems for all $\widetilde{\mathbf{p}_k}$ substituting $\mathbf{q}$ and λ_{ij} by $\mathbf{q}_k$ and λ_{ij}^k. It can be shown that the sequence of Kuhn–Tucker multipliers $(\boldsymbol{\lambda}^k)$ is bounded. The rest of the proof relies on standard compactness methods. □

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