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Calculus of Variations/Mathematical Problems in Mechanics

Local minimizers of one-dimensional variational problems and obstacle problems ☆

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Abstract

In this Note we suggest a direct approach to study local minimizers of one-dimensional variational problems. ***To cite this article: M.A. Sychev, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Minimiseurs locaux de problèmes variationnels en une dimension et de problémes d'obstacle. Dans cette Note nous suggérons une approche directe pour étudier les minimiseurs locaux de problèmes variationnels monodimensionnels. ***Pour citer cet article : M.A. Sychev, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

On considère un problème de minimisation de la forme (1), (2). Nous considérons en premier lieu le problème d'obstacle (3). On suppose que la fonction L est convexe en v, et satisfait les hypothèses (H1) et (H2). Nos résultats concernant le problème d'obstacle sont les deux théorèmes suivants :

Théorème 1. *Soit L vérifiant* (H1)–(H2) *et soient $c, \epsilon > 0$. Supposons $f > g$ tels que $\|f\|_{W^{1,\infty}}, \|g\|_{W^{1,\infty}} \leqslant c$.*

Alors il existe $\delta = \delta(\epsilon) > 0$ tel que le problème (1)–(3) *avec $f(a) \geqslant A \geqslant g(a)$, $f(b) \geqslant B \geqslant g(b)$ ait une solution dans $W^{1,\infty}[a,b]$ dès que $\|f - g\|_C \leqslant \delta$. De plus, chaque solution u vérifie*

$$\|\dot{u}\|_{L^\infty} \leqslant \max\{\|\dot{f}\|_{L^\infty}, \|\dot{g}\|_{L^\infty}\} + \epsilon.$$

Une plus grande regularité de f, g (C^1 ou $C^{1,\alpha}$) se reporte sur u (C^1 ou $C^{1,\gamma}$). Ce résultat est uniforme en les fonctions L qui satisfont (H1)–(H2)*, pourvu que ces hypothèses soient verifiées uniformément dans l'ensemble $K_\nu = [a,b] \times [-c-\nu, c+\nu]^2$ pour un certain $\nu > 0$.*

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Ce théorème peut-être précisé lorsque L est C^3, voir Theorem 5.

Théorème 2. *Soit L satisfaisant* (H1)–(H2) *et soient $c, \alpha > 0$. On suppose $\|f\|_{C^{1,\alpha}}, \|g\|_{C^{1,\alpha}} \leqslant c$, $f > g$. Il existe $\delta > 0$ et $\bar{c}, \gamma > 0$ tels que chaque problème* (1)–(3) *où $f(a) \geqslant A \geqslant g(a)$, $f(b) \geqslant B \geqslant g(b)$ ait une solution C^1 pourvu que $\|f - g\|_C \leqslant \delta$. De plus, $\|u\|_{C^{1,\gamma}} \leqslant \bar{c}$. Ce résultat est uniforme en les fonctions L satisfaisant* (H1)–(H2) *uniformément dans un ν-voisinage, $\nu > 0$, d'un compact $K \subset \mathbf{R}^3$ contenant les graphes des fonctions f, g.*

Une première application du Théorème 2 est la correspondance entre minimiseurs locaux faibles et forts.

Théorème 3. *Si L satisfait* (H1)–(H2)*, tout minimiseur local faible est $C^{1,\alpha}$ pour un $\alpha > 0$ adéquat et est un minimiseur local fort. De plus tout minimiseur local faible isolé est fort.*

Une autre application du Théorème 2 est la suivante :

Théorème 4. *Soit L satisfaisant* (H1)–(H2) *et u un minimiseur local isolé de* (1), (2)*. Supposons que L_n vérifie* (H1)–(H2) *uniformément en n dans un voisinage V_ϵ du graphe de u (voir Theorem 4). Si $\|L_n - L\|_{C(V_\epsilon)} \to 0$ alors il existe des minimiseurs locaux u_n des problèmes* (1), (2)$_n$ *tels que $\|u_n - u\|_{C^1} \to 0$.*

Dans le cas $L \in C^3$, la théorie classique montre sous des hypothèses de convexité adéquate que l'équation d'Euler–Lagrange est vérifiée et implique (5). On montre finalement l'alternative suivante :

Théorème 5. *Soit $L \in C^3$ tel que $L_{vv} > 0$ et u un minimiseur local de* (1), (2)*. La situation est l'une de celles décrite ci-dessous* :

(i) *il existe des solutions u_n^+, u_n^- de* (5) *telles que $u_n^+ > u > u_n^-$, avec u_n^+ décroissante en n, u_n^- croissante en n, telles que $\|u_n^+ - u\|_{C^1}, \|u_n^- - u\|_{C^1} \to 0$ lorsque $n \to \infty$* ;
(ii) *Pour un certain $\epsilon > 0$, toutes les solutions des problèmes de Cauchy $w(a) = A$, $\dot{w}(a) = \alpha$, $\alpha \in [\dot{u}(a), \dot{u}(a) + \epsilon]$ de* (5) *vérifient à la fois* (2) *et l'égalité $J(w) = J(u)$, et il existe une suite croissante u_n^- de solutions de* (5) *telles que $u_n^- < u$ et $\|u_n^- - u\|_{C^1} \to 0$* ;
(iii) *Pour un certain $\epsilon > 0$, toutes les solutions des problèmes de Cauchy $w(a) = A$, $\dot{w}(a) = \alpha$, $\alpha \in [\dot{u}(a) - \epsilon, \dot{u}(a)]$ de* (5) *satisfont à la fois* (2) *et l'egalité $J(w) = J(u)$, et il existe une suite décroissante u_n^+ de solution de* (5) *telles que $u_n^+ > u$ et $\|u_n^+ - u\|_{C^1} \to 0$* ;
(iv) *Pour un certain $\epsilon > 0$, toutes les solutions des problèmes de Cauchy $w(a) = A$, $\dot{w}(a) = \alpha$, $\alpha \in [\dot{u}(a) - \epsilon, \dot{u}(a) + \epsilon]$ de* (5) *satisfont à la fois* (2) *et l'identité $J(w) = J(u)$.*

1. Introduction

Consider a minimization problem

$$J(u) = \int_a^b L\big(x, u(x), \dot{u}(x)\big)\,\mathrm{d}x \to \min, \quad L(x, u, v) : [a, b] \times \mathbf{R}^2 \to \mathbf{R}, \tag{1}$$

$$u(a) = A, \quad u(b) = B, \quad u : [a, b] \to \mathbf{R}. \tag{2}$$

Recall that $u \in C^1[a, b]$ is called a weak local minimizer (an isolated weak local minimizer) of the problem (1), (2) if (2) holds and there exists $\epsilon > 0$ such that $J(u) \leqslant J(w)$ $(J(u) < J(w))$ for any $w \in C^1[a, b]$ that satisfies (2), $w \neq u$ and $\|u - w\|_{C^1[a,b]} \leqslant \epsilon$. A function $u \in C^1[a, b]$ is called a strong local minimizer (an isolated strong local minimizer) of the problem (1), (2) if (2) holds and for some $\epsilon > 0$ we can guarantee that $J(u) \leqslant J(w)$ $(J(u) < J(w))$ for all $w \in C^1[a, b]$ such that w satisfies (2), $w \neq u$ and $\|u - w\|_{C[a,b]} \leqslant \epsilon$. Any strong local minimizer is obviously a weak local minimizer. Strong local minimizers are solutions of the problem (1)–(3) in the class C^1, where

$$g \leqslant u \leqslant f, \quad f, g : [a, b] \to \mathbf{R}, \tag{3}$$

for appropriate f, g, say for $g = u - \epsilon$, $f = u + \epsilon$ with $\epsilon > 0$ sufficiently small.

We emphasize that in this paper weak and strong local minimizers are C^1-regular functions.

The problems (1)–(3) are known as obstacle problems and their theory repeat the general existence and regularity theory for Sobolev solutions of the problem (1), (2), see e.g. [3]. Therefore it requires a number of assumptions on growth of the first and second derivatives of L. It turns out that the case of thin strips, i.e. when $\|f-g\|_C$ is sufficiently small, admits a better solvability theory which involves only standard assumptions of the classical local theory, i.e. certain requirements on regularity of L and on ellipticity in v. We will recall the results of this theory below.

We assume

$$L(x,u,v):[a,b]\times \mathbf{R}^2\to \mathbf{R} \text{ is locally Holder continuous,} \tag{H1}$$

L is convex in v and for each compact subset K of $\mathbf{R}^3$ there exists $\mu=\mu(K)>0$ such that

$$L(x,u,v_2)-L(x,u,v_1)-l(v_2-v_1)\geqslant \mu(v_2-v_1)^2 \tag{H2}$$

provided $(x,u,v_1),(x,u,v_2)\in K$, $l\in\partial_v L(x,u,v_1)$, where $\partial_v L(x,u,v_1)$ is the subgradient of the function $L(x,u,\cdot):\mathbf{R}\to\mathbf{R}$ at v_1.

The following two theorems state our results for the obstacle problem (1)–(3).

Theorem 1 *(an a priori estimate). Let L satisfy* (H1)–(H2) *and let $c,\epsilon>0$. Let also $f,g:[a,b]\to\mathbf{R}$ be such that $\|f\|_{W^{1,\infty}},\|g\|_{W^{1,\infty}}\leqslant c$, $f>g$ in $]a,b[$.*

There exists $\delta=\delta(\epsilon)>0$ such that each such problem (1)–(3) *with $f(a)\geqslant A\geqslant g(a)$, $f(b)\geqslant B\geqslant g(b)$ has a solution in the class $W^{1,\infty}[a,b]$ provided $\|f-g\|_C\leqslant\delta$, moreover for each such solution u we have*

$$\|\dot u\|_{L^\infty}\leqslant \max\{\|\dot f\|_{L^\infty},\|\dot g\|_{L^\infty}\}+\epsilon;$$

in case $f,g\in C^1$ the solutions are also C^1-regular, in case $f,g\in C^{1,\alpha}$ we can assert that $u\in C^{1,\gamma}$ for some $\gamma>0$.

The result is uniform with respect to L that satisfy (H1)–(H2) *provided the assumptions* (H1)–(H2) *hold uniformly in the set $K_\nu=[a,b]\times[-c-\nu,c+\nu]^2$ for certain $\nu>0$.*

Theorem 2 *(compactness in C^1). Let L satisfy* (H1)–(H2) *and let $c,\alpha>0$. We assume $\|f\|_{C^{1,\alpha}},\|g\|_{C^{1,\alpha}}\leqslant c$, $f>g$ in $]a,b[$.*

There exists $\delta>0$ and $\bar c,\gamma>0$ such that each such problem (1)–(3) *with $f(a)\geqslant A\geqslant g(a)$, $f(b)\geqslant B\geqslant g(b)$ is solvable in the class C^1 provided $\|f-g\|_C\leqslant\delta$, moreover each solution u satisfies the inequality $\|u\|_{C^{1,\gamma}}\leqslant\bar c$.*

The result is uniform with respect to L that satisfy (H1)–(H2) *provided* (H1)–(H2) *hold uniformly in a ν-neighbourhood, $\nu>0$, of a compact set $K\subset\mathbf{R}^3$ which contains graphs of the functions f,g.*

We can use these results to study a number of open problems for local minimizers and to complete this way the classical local theory, which results we recall for convenience.

2. Applications of Theorems 1 and 2

Recall the standard results of the classical local theory for the problems (1), (2), cf. e.g. [3].

In case u is a weak local minimizer and in case $L\in C^1$ the Euler–Lagrange equation holds

$$\frac{\mathrm{d}}{\mathrm{d}x}L_v\big(x,u(x),\dot u(x)\big)=L_u\big(x,u(x),\dot u(x)\big). \tag{4}$$

If $L\in C^2$, $L_{vv}>0$ then $u\in C^2$ and differentiating (4) we obtain

$$\ddot u=\frac{L_u-L_{xv}-L_{uv}\dot u}{L_{vv}}, \tag{5}$$

see [4] for precise results on necessity of convexity in v.

The next necessary requirement for u to be a weak local minimizer is the Jacobi condition

$$u_\alpha(x,\alpha_0)>0,\quad x\in]a,b[,\ \alpha_0=\dot u(a), \tag{6}$$

where for each α the function $u(\cdot,\alpha)$ solves (5) with $u(a,\alpha)=A$, $\dot u(a,\alpha)=\alpha$; here we need to assume that $L\in C^3$.

In case

$$u_\alpha(x,\alpha_0) > 0, \quad x \in]a,b],\ \alpha_0 = \dot{u}(a) \tag{7}$$

Weierstrass has proved that u is a strong local minimizer. In this case

$$u_\alpha(x,\alpha_0) > 0, \quad x \in]a-\epsilon,b],$$

for a sufficiently small $\epsilon > 0$ and for all α_0 close to $\dot{u}(a-\epsilon)$; here $u(a-\epsilon,\alpha) = u(a-\epsilon)$, $\dot{u}(a-\epsilon,\alpha) = \alpha$. The graphs of solutions of such Cauchy problems for (5) form a neighbourhood U of the graph of u in $[a,b]$ and do not intersect. Therefore for each $(x,u) \in U$ we can find α such that $u(a-\epsilon,\alpha) = u(a-\epsilon)$, $u(x,\alpha) = u$ and we can define

$$\Phi(x,u) := \int_{a-\epsilon}^{x} L\big(y, u(y,\alpha), \dot{u}(y,\alpha)\big)\,\mathrm{d}y. \tag{8}$$

A marvelous observation by Weierstrass says that

$$L(x,u,v) \geqslant \Phi_x(x,u) + \Phi_u(x,u)v, \quad (x,u) \in U,\ v \in \mathbf{R}, \tag{9}$$

where the equality holds only for $v = \dot{u}(x,\alpha)$.

Therefore $J(u) = \Phi(b,B) - \Phi(a,A)$ and $J(w) > \Phi(b,B) - \Phi(a,A)$ for any other function $w \in C^1$ that satisfies (2) with the graph lying in U.

Further developments were connected with so-called generalized solutions, i.e. solutions in the class of Sobolev functions. What could be proved is that the functional $J(u)$ is lower semicontinuous with respect to weak convergence in $W^{1,1}$ provided $L(x,u,v)$ is convex in v. Then the superlinear growth of L in v implies weak compactness of the minimizing sequences and then the existence of a global minimizer. Ellipticity $L_{vv} > 0$ implies partial regularity of these solutions, i.e. $u(\cdot)$ has classical derivative, finite or infinite, everywhere and the derivative $\dot{u}(\cdot)$ is continuous as function with values in $\bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$. In particular $u(\cdot)$ satisfies the Euler equation (5) in an open set of full measure. Therefore full regularity of solutions follows provided there no solutions of the Euler equation bounded in C-norm with unbounded derivative. There results are due to Tonelli and his theory remained unchanged until recent days, cf. [3].

However this theory of generalized solution is insufficient to deal with the obstacle problems in generality required and we need to develop another direct approach. Moreover Theorems 1, 2 allow to resolve a number of questions, stated below, for local minimizers that could not be treated in context of the classical local theory or of Tonelli's direct arguments. This way, in addition to independent interest from the point of view of obstacle problems Theorems 1, 2 suggest a direct approach to attack general unconstrained problems.

Question 1. Is each weak local minimizer also a strong local minimizer?

This question was stated recently by Brezis and Nirenberg in [2]. When in the multi-dimensional case the question was addressed for special classes of integrands L, see [8,9] in the one-dimensional case one can hope to resolve it under the standard assumptions of the classical local theory. In fact, the Weierstrass result says that this is true in case (7). The only open case if (6). However nothing is known if $L \notin C^3$.

We can apply Theorem 2 with $f = u + \epsilon$, $g = u - \epsilon$ to obtain

Theorem 3. *Let L satisfy* (H1)–(H2). *Then each weak local minimizer is $C^{1,\alpha}$-regular for an appropriate $\alpha > 0$ and is a strong local minimizer. Moreover each isolated weak local minimizer is an isolated strong local minimizer.*

Therefore everywhere below we do not specify the cases of weak and strong minimizers saying simply "a local minimizer".

Question 2. Let u be a local minimizer for the problem (1), (2) with $L \in C^\infty$, $L_{vv} > 0$. Let $L_n \in C^\infty$, $(L_n)_{vv} > 0$ and let $\|L_n - L\|_C \to 0$. Is this correct that there exist local minimizers u_n associated with L_n such that $\|u_n - u\|_{C_1} \to 0$.

This question belongs to Ulam [10, Ch. 6, §1] who remarked that to apply the classical local theory we need $\|L_n - L\|_{C^3} \to 0$, when the variational problem (1), (2) is posed only by L, but the derivatives.

Again due to Theorem 2 we can assert

Theorem 4. *Let L satisfy* (H1)–(H2) *and let u be an isolated local minimizer for* (1), (2). *Let L_n also satisfy* (H1)–(H2)*, moreover let L_n satisfy* (H1)–(H2) *uniformly with respect to n in the set V_ϵ for some $\epsilon > 0$, where*

$$V_\epsilon := \{(x, u, v)\colon\ x \in [a, b],\ |u - u(x)| \leqslant \epsilon,\ |v - \dot{u}(x)| \leqslant \epsilon\}.$$

If $\|L_n - L\|_{C(V_\epsilon)} \to 0$ then there exist local minimizers u_n of the problem (1), $(2)_n$ *such that $\|u_n - u\|_{C^1} \to 0$.*

Remark. The result remains valid for a more general case when u is no longer an isolated minimizer provided there exist positive numbers $\epsilon_n^+ \to 0$, $\epsilon_n^- \to 0$ such that the functions $u + \epsilon_n^+$, $u - \epsilon_n^-$, $n \in \mathbf{N}$, do not touch other minimizers of the problem.

Question 3. What are necessary and sufficient conditions for the problem (1), (2) admit a local minimizer?

The classical local theory indicates that (7) is a sufficient condition for u to be a local minimizer and involves requirements on a family of solutions of (5) of one parameter. One could look for a less demanding condition that includes also the situation (6).

Theorem 1 can be precised in case $L \in C^3$.

Theorem 5. *Let $L \in C^3$, $L_{vv} > 0$. Assume f and g to be such solutions of* (5) *that $f > g$ in $[a, b]$, $f(a) > A > g(a)$, $f(b) > B > g(B)$ and $\|f - g\|_C < 1/8M$, where*

$$M := \max\{|F(x, u, v)|\colon\ x \in [a, b],\ g(x) \leqslant u \leqslant f(x),\ |v| \leqslant \max\{\|\dot{f}\|_C, \|\dot{g}\|_C\} + 1\}$$

with F being the right-hand side of (5).

Then there exists a local minimizer u of (1), (2) *such that $g < u < f$ in $[a, b]$.*

Theorem 5 uses in fact the necessary condition which excludes only the case when u could be embedded in a family of minimizers of one parameter.

Theorem 6. *Let $L \in C^3$, $L_{vv} > 0$ and let u be a local minimizer for the problem* (1), (2). *Then one the following situations hold*:

1) *there exist solutions u_n^+, u_n^- of* (5) *such that $u_n^+ > u$, $u_n^- < u$ in $[a, b]$ and u_n^+ is a decreasing sequence, u_n^- is an increasing sequence with $\|u_n^+ - u\|_{C^1}, \|u_n^- - u\|_{C^1} \to 0$ as $n \to \infty$;*
2) *for certain $\epsilon > 0$ all solutions of the Cauchy problems*

   $$w(a) = A, \quad \dot{w}(a) = \alpha, \quad \alpha \in [\dot{u}(a), \dot{u}(a) + \epsilon]$$

   for (5) *satisfy both* (2) *and the equality $J(w) = J(u)$, there exists an increasing sequence u_n^- of solutions of* (5) *such that $u_n^- < u$ and $\|u_n^- - u\|_{C^1} \to 0$;*
3) *for certain $\epsilon > 0$ all solutions of the Cauchy problems*

   $$w(a) = A, \quad \dot{w}(a) = \alpha, \quad \alpha \in [\dot{u}(a) - \epsilon, \dot{u}(a)]$$

   for (5) *satisfy both* (2) *and the equality $J(w) = J(u)$, there exists a decreasing sequence u_n^+ of solutions of* (5) *such that $u_n^+ > u$ and $\|u_n^+ - u\|_{C^1} \to 0$;*
4) *for certain $\epsilon > 0$ all solutions of the Cauchy problems*

   $$w(a) = A, \quad \dot{w}(a) = \alpha, \quad \alpha \in [\dot{u}(a) - \epsilon, \dot{u}(a) + \epsilon]$$

   for (5) *satisfy both* (2) *and the equality $J(w) = J(u)$.*

Remark. In the situation 2 there exists $\delta > 0$ such that there are no solution u^+ of (5) with $u^+ > u$ in $[a, b]$, $\|u^+ - u\|_C \leqslant \delta$.

Method of the proofs of Theorems 1, 2 develops an elementary direct approach suggested in [6]; however the proofs require an independent scheme. Therefore Theorem 5 presents also a direct proof of sufficiency of (7) for the solution u of (5) to be a local minimizer. The results of Theorems 1, 2 fail in the case of thick strips since there are no solvability theorem in $W^{1,\infty}$ or in C^1 for problems (1), (2) even in the case $L(x,u,v) \geqslant |v|^2$, $L_{vv} > 0$, in which a Sobolev solution always exists. First examples when Sobolev solutions fail to be $W^{1,\infty}$-regular were suggested in [1]. In [7] we constructed examples of problems (1), (2) without solutions in the class $C^1\,]a,b[\,\cap C[a,b]$ even for the case $L \in C^\infty$, $\mu_1|v|^2 \leqslant L \leqslant \mu_2|v|^2$, $0 < \mu_1 \leqslant L_{vv} \leqslant \mu_2$. The paper [5] indicates a necessary and sufficient condition for the Sobolev solutions to be $W^{1,\infty}$, and then C^1-regular, though it is not that effective.

The multi-dimensional analogue of Theorems 1, 2, i.e. the a priori estimate and compactness in C^1 of solutions of the minimization problems in thin strips, is an important and difficult open problem.

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