Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 995–998
COMPTES RENDUS MATHEMATIQUE
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Dynamical Systems

About a low complexity class of cellular automata

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Received 21 November 2007; accepted after revision 18 July 2008

Available online 22 August 2008

Presented by Étienne Ghys

Abstract

Extending to all probability measures the notion of μ-equicontinuous cellular automata introduced for Bernoulli measures by Gilman, we show that the entropy is null if μ is an invariant measure and that the sequence of image measures of a shift ergodic measure by iterations of such automata converges in Cesàro mean to an invariant measure μ_c. Moreover, this cellular automaton is still μ_c-equicontinuous and the set of periodic points is dense in the topological support of the measure μ_c. The last property is also true when μ is invariant and shift ergodic. ***To cite this article: P. Tisseur, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur une classe d'automate cellulaire de faible complexitée. Nous étendons à toute mesure de probabilité, la notion d'automate cellulaire μ-equicontinus introduit en premier lieu pour des mesures de Bernoulli par Gilman et nous montrons que l'entropie de l'automate est nulle si μ est invariante mais aussi que la suite des mesures images d'une mesure ergodique pour le décalage converge en moyenne de Cesàro vers une mesure invariante notée μ_c. De plus, cet automate cellulaire a encore la particularité d'être μ_c-equicontinu et l'ensemble des points périodiques est dense dans le support topologique de la mesure μ_c. Cette dernière propriété est aussi vraie pour cette classe d'automate si la mesure μ est invariante et shift ergodique. ***Pour citer cet article : P. Tisseur, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction, definitions

Let A be a finite set. We denote by $A^{\mathbb{Z}}$, the set of bi-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ where $x_i \in A$. We endow $A^{\mathbb{Z}}$ with the product topology of the discrete topologies on A. A point $x \in A^{\mathbb{Z}}$ is called a configuration. The shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by: $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$. A cellular automaton (CA) is a continuous self-map F on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis–Hedlund–Lyndon theorem states that for every cellular automaton F there exist an integer r and a block map f from A^{2r+1} to A such that $F(x)_i = f(x_{i-r}, \ldots, x_i, \ldots, x_{i+r})$. The integer r is called the radius of the cellular automaton. For integers i, j with $i \leqslant j$ we denote by $x(i, j)$ the word $x_i \ldots x_j$ and by $x(i,\infty)$ the infinite sequence $(v_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ one has $v_n = x_{i+n}$. For any integer $n \geqslant 0$ and point $x \in A^{\mathbb{Z}}$, we denote by $B_n(x)$ the set of points y such that for all $i \in \mathbb{N}$, one has $F^i(x)(-n, n) = F^i(y)(-n, n)$ and by $C_n(x)$ the set of points

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doi:10.1016/j.crma.2008.07.018

y such that $y_j = x_j$ with $-n \leqslant j \leqslant n$. A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point if, for all positive integers n, there exists another positive integer m such that $B_n(x) \supset C_m(x)$. A point x is μ-equicontinuous if for all $m \in \mathbb{N}$ one has

$$\lim_{n\to\infty} \frac{\mu(C_n(x) \cap B_m(x))}{\mu(C_n(x))} = 1.$$

In this Note, we call μ-equicontinuous CA any cellular automaton with a set of full measure of μ-equicontinuous points. Clearly an equicontinuous point which belongs to

$$S(\mu) = \overline{\left\{x \in A^{\mathbb{Z}} \mid \mu\big(C_n(x)\big) > 0 \mid \forall n \in \mathbb{N}\right\}},$$

(the topological support of μ) is also a μ-equicontinuous point. When μ is a shift ergodic measure, the existence of μ-equicontinuous points implies than the cellular automaton is μ-equicontinuous (see [2]).

These definitions was motivated by the work of Wolfram (see [6]) who proposed a first empirical classification based on computer simulations. In [2] Gilman introduced a formal and measurable classification by dividing the set of CA in three parts (CA with equicontinuous points, CA without equicontinuous points but with μ-equicontinuous points, μ-expansive CA). Gilman's classes are defined thanks to a Bernoulli measure, not necessarily invariant, and corresponds to the Wolfram's simulations based on random entry. Here we study some properties of the μ-equicontinuous class that allows one to construct easily invariant measures (see Theorem 5) and we try to describe what kind of dynamic characterizes μ-equicontinuous CA when μ is an invariant measure. Finally, remark that the comparison between equicontinuity (see some properties of this class in [1] and [4]) and μ-equicontinuity make more sense when we study the restriction of the automaton to $S(\mu)$ (see Section 4 for comments and examples).

2. Statement of the results

2.1. Gilman's results

Proposition 1. *(See [3].) If $\exists x$ and $m \neq 0$ such that $B_n(x) \cap \sigma^{-m} B_n(x) \neq \emptyset$ with $n \geqslant r$ (the radius of the automaton F) then the common sequence $(F^i(y)(-n, n))_{i\in\mathbb{N}}$ of all points $y \in B_n(x)$ is ultimately periodic.*

In [3] Gilman states the following result for any Bernoulli measure μ. The proof uses only the shift ergodicity of these measures and can be extended to any shift ergodic measure.

Proposition 2. *(See [3].) Let μ be a shift ergodic measure. If a cellular automaton F has a μ-equicontinuous point, then for all $\epsilon > 0$ there exists a F-invariant closed set Y such that $\mu(Y) > 1 - \epsilon$, and the restriction of F to Y is equicontinuous.*

2.2. New results

Proposition 3. *The measure entropy $h_\mu(F)$ of a μ-equicontinuous and μ-invariant cellular automaton F (with μ not necessarily shift invariant) is equal to zero.*

Proposition 4. *If a cellular automaton F has some μ-equicontinuous points where μ is a F-invariant and shift ergodic measure then the set of F-periodic points is dense in the topological support of μ.*

Theorem 5. *Let μ be a shift-ergodic measure. If a cellular automaton F has some μ-equicontinuous points, then the sequence*

$$(\mu_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\sum_{i=0}^{n-1} \mu \circ F^{-i}\right)_{n\in\mathbb{N}}$$

converges vaguely to an invariant measure μ_c.

Theorem 6. *If μ is a shift ergodic measure and F a μ-equicontinuous cellular automaton then F is also a μ_c-equicontinuous cellular automaton.*

Corollary 7. *If $\mu_c = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu \circ F^{-i}$ where μ is a shift ergodic measure and F is a cellular automaton with μ-equicontinuous points then the set of F-periodic points is dense in $S(\mu_c)$.*

3. Sketches of the proofs

3.1. Proof of Proposition 3

Denote by $(\alpha_p)_{p\in\mathbb{N}}$ the partition of $A^{\mathbb{Z}}$ by the $2p+1$ central coordinates and remark that

$$h_\mu(F) = \lim_{p\to\infty} h_\mu(F, \alpha_p)$$

where $h_\mu(F,\alpha_p)$ denotes the measurable entropy with respect to the partition α_p. Using the Shannon–McMillan–Breiman Theorem, we can show that $\forall p \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$h_\mu(F,\alpha_p) \leqslant \int \lim_{n\to\infty} \frac{-\log \mu(B_m(x))}{n}\,\mathrm{d}\mu(x) = 0.$$

3.2. Proof of Theorem 5

It is sufficient to show that for all $x \in S(\mu)$ and $m \in \mathbb{N}$ the sequence $(\mu_n(C_m(x)))_{n\in\mathbb{N}}$ converges. From Proposition 2 there exists a set Y_ϵ of measure greater than $1-\epsilon$ such that for all points $y \in Y_\epsilon$ and positive integer k the sequences $(F^n(y)(-k,k))_{n\in\mathbb{N}}$ are eventually periodic with preperiod $pp_\epsilon(k)$ and period $p_\epsilon(k)$. We get that $\mu_n(C_m(x)\cap Y_\epsilon) = \frac{1}{n}\sum_{i=0}^{pp_\epsilon(k)-1} \mu(F^{-i}(C_m(x)) \cap Y_\epsilon) + \frac{1}{n}\sum_{i=pp_\epsilon(k)}^{n-1} \mu(F^{-i}(C_m(x))\cap Y_\epsilon)$ for all $x \in A^{\mathbb{Z}}$ and integer $k \geqslant m$. Remark that the first term tends to 0 and the periodicity of the second one implies that $\lim_{n\to\infty} \mu_n(C_m(x) \cap Y_\epsilon) = \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu(F^{-(i+pp_\epsilon(k))}(C_m(x)\cap Y_\epsilon))$. Moreover, we have $\lim_{\epsilon\to 0} \mu_n(C_m(x)\cap Y_\epsilon) = \mu_n(C_m(x))$. Since for all x and $m \in \mathbb{N}$ one has $|\mu_n(C_m(x)\cap Y_\epsilon) - \mu_n(C_m(x))| \leqslant \frac{n\epsilon}{n} = \epsilon$ the convergence is uniform with respect to ϵ. It follows that we can reverse the limits and obtain that

$$\begin{aligned}\mu_c &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu\circ F^{-i}\big(C_m(x)\big) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} \lim_{\epsilon\to 0} \mu \circ F^{-i}\big(C_m(x)\cap Y_\epsilon\big)\\ &= \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\circ F^{-i}\big(C_m(x)\cap Y_\epsilon\big) = \lim_{\epsilon\to 0}\frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu\big(F^{-(i+pp_\epsilon(k))}\big(C_m(x)\big)\cap Y_\epsilon\big) = \mu_c\big(C_m(x)\big).\end{aligned}$$

The invariance of converging subsequences of $(\mu_n)_{n\in\mathbb{N}}$ is a classical result.

3.3. Proof of Proposition 4

Since μ is a shift ergodic measure and there exist a μ-equicontinuous points x, for all $m \in \mathbb{N}$ and $z \in S(\mu)$ there exist $(i,j) \in \mathbb{N}^2$ such that $\mu(C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x) =: S) > 0$ (r is the radius of the CA). From the Poincaré recurrence theorem, for all $z \in S(\mu)$, there exists $m \in \mathbb{N}$ and $y \in S$ such that $F^m(y)(-r-p-i, j+p-r-1) = y(-r-p-i, j+p-r-1)$. From the Proof of Proposition 1 (see [3]), the shift periodic point $\overline{w} = \dots www\dots$ such that $\overline{w}(-r-p-i, j+p-r-1) = w = y(-r-p-i, j+p-r-1)$ belongs to S and since the F orbit of each $y' \in S \cap \{y'' \in A^{\mathbb{Z}} \mid y''_l = y_l \mid (-r-p-i \leqslant l \leqslant j+p-r-1)\}$ share the same central coordinates, it follows that $F^m(\overline{w})(-r-p-i, j+p-r-1) = w = \overline{w}(-r-p-i, j+p-r-1)$ which implies that $F^m(\overline{w}) = \overline{w}$ and permit to conclude.

3.4. Proof of Theorem 6 and Corollary 7

Let x be a μ-equicontinuous point. For all $m \in \mathbb{N}$, define $Y_m := \bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(x) \cap \sigma^{j+m}B_r(x))$ (r is the radius of F) and $\Omega_m = \lim_{n\to\infty}\bigcap_{j=0}^{n}\bigcup_{i=j}^{\infty}F^i(Y_m)$ (the omega-limit set of Y_m under F). Since μ is a shift ergodic measure and $\mu(B_r(x)) > 0$, for all $m \in \mathbb{N}$, we get that $\mu(Y_m) = 1$ and consequently $\mu_c(\Omega_m) = 1$. Let $\Lambda(F)$ be the omega-limit set of $A^{\mathbb{Z}}$. Using the eventual periodicity of $(F^n(x)(-r,r))_{n\in\mathbb{N}}$ (see Proposition 1), it can be proved that the omega-limit set of $B_r(x)$ is a finite union of sets $B_r(z_l) \cap \Lambda(F)$ ($0 \leqslant l \leqslant p-1$). This implies that $\Omega_m = \bigcup_{z\in[z_0\ldots z_{p-1}]}\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(z) \cap \sigma^{j+m}B_r(z)) \cap \Lambda(F)$ and it follows that for all $z \in S(\mu_c)$ and $k \in \mathbb{N}$, the inequality $\mu_c(C_k(z) \cap \Omega_k) > 0$ implies that there always exist a point z' and integers $i, j \geqslant m$ such that $\mu_c(C_p(z) \cap \sigma^{-(i+p)}B_r(z') \cap \sigma^{j+p}B_r(z')) > 0$. Using final arguments of the proof of Proposition 4, the last inequality is sufficient to show Corollary 7. For any measurable set E, define $E^{\mu_c} = \{y \in E \mid \lim_{n\to\infty}\frac{\mu_c(C_n(y)\cap E)}{\mu_c(C_n(y))} = 1\}$. For all $m \in \mathbb{N}$, define $\Omega'_m := \bigcup_{z\in[z_0\ldots z_{p-1}]}\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(z) \cap \sigma^{j+m}B_r(z))^{\mu_c} \cap \Lambda(F)$ and denote by Ω the set $\bigcap_{m\in\mathbb{N}}\Omega'_m$. Since for all measurable set E, one has $\mu_c(E^{\mu_c}) = \mu_c(E)$, for all $m \in \mathbb{N}$, we get that $\mu_c(\Omega'_m) = 1$ and consequently $\mu_c(\Omega) = 1$. Since for all $y \in \Omega$ and $k \in \mathbb{N}$ there exist integers $i, j \geqslant k$ and a point z' such that $y \in \sigma^{-i}(B_r(z') \cap \sigma^j B_r(z'))^{\mu_c}$, we obtain that $y \in B_m^{\mu_c}(y)$ which finishes the proof.

4. Example of μ-equicontinuous CA without equicontinuous points

In [2] Gilman gives an example of a μ-equicontinuous CA F_s that has no equicontinuous points. The automaton F_s act on $\{0,1,2\}^{\mathbb{Z}}$ and is defined thanks to the following block map of radius 1:

∗00	∗01	∗02	∗10	∗11	∗12	∗20	∗21	∗22
0	1	0	0	1	0	2	0	2

The letter ∗ stands for any letter in $\{0,1,2\}$. Considering 0 as a background element, the 2's move straight down, 1's move to the left and 1 and 2 collide annihilate each other. In this case the measure μ is a Bernoulli measure on $\{0,1,2\}^{\mathbb{Z}}$ and the existence of μ-equicontinuous points depends on the parameters $p(0)$, $p(1)$, $p(2)$ of this measure. In [2] it is shown that if $p(2) > p(1)$ then the probability that a 2 is never annihilated is positive and this implies that there exist μ-equicontinuous points. Since the existence or non-existence of a sufficient number of 1 in the right side can always modify the central coordinates one has $C_m(x) \not\subset B_n(x)$ for all $n, m \in \mathbb{N}$ which implies that there is no equicontinuous points.

Note that using Theorems 5 and 6 the automaton F_s is μ_c-equicontinuous if $p(2) > p(1)$ but the restriction of F_s to $S(\mu_c)$ always has equicontinuous points ($S(\mu_c) = \{0,2\}^{\mathbb{N}}$ and $F : S(\mu_c) \to S(\mu_c)$ is the identity). In [5], we describe a more complex CA $\mathcal{F}$ such that $\mathcal{F} : S(\mu_c) \to S(\mu_c)$ is μ_c-equicontinuous, without equicontinuous points and the invariant measure μ_c is construct thanks to Theorem 5.

Acknowledgements

We wish to acknowledge the support of the CNPq, the Departamento de Matemática da UNESP, São José do Rio Preto and the CMCC da UFABC, Santo André, Brasil in which this work have been done and the referee for his suggestions.

References

[1] F. Blanchard, P. Tisseur, Some properties of cellular automata with equicontinuity points, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 36 (5) (2000) 569–582.
[2] R.H. Gilman, Classes of linear automata, Ergodic Theory and Dynamical Systems 7 (1987) 105–118.
[3] R.H. Gilman, Periodic behaviour of linear automata, in: Dynamical Systems, in: Lecture Notes in Mathematics, vol. 1342, Springer, New York, 1988, pp. 216–219.
[4] P. Tisseur, Cellular automata and Lyapunov exponents, Nonlinearity 13 (2000) 1547–1560.
[5] P. Tisseur, Density of periodic points, invariant measures and almost equicontinuous points of cellular automata, Preprint.
[6] S. Wolfram, Theory and Applications of Cellular Automata, World Scientific, 1986.