Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 346 (2008) 653–656

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Differential Geometry

Complete gradient shrinking Ricci solitons have finite topological type

Fu-quan Fang [a], Jian-wen Man [b], Zhen-lei Zhang [b]

[a] *Department of Mathematics, Capital Normal University, Beijing, PR China*
[b] *Chern Institute of Mathematics, Weijin Road 94, Tianjin 300071, PR China*

Received 3 January 2008; accepted after revision 18 March 2008

Available online 29 April 2008

Presented by Étienne Ghys

Abstract

We show that a complete Riemannian manifold has finite topological type (i.e., homeomorphic to the interior of a compact manifold with boundary), provided its Bakry–Émery Ricci tensor has a positive lower bound, and either of the following conditions:
(i) the Ricci curvature is bounded from above;
(ii) the Ricci curvature is bounded from below and injectivity radius is bounded away from zero.
Moreover, a complete shrinking Ricci soliton has finite topological type if its scalar curvature is bounded. ***To cite this article: F.-q. Fang et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Les solitons de Ricci contractants complets sont de type fini. Dans cette Note nous montrons qu'une variété riemanienne complète est de type topologique fini – c'est-à-dire qu'elle est homéomorphe à une variété compacte à bord – si son tenseur de Bakry–Emery–Ricci est bornée inférieurement par une constante positive et vérifie l'une des conditions suivantes :
(i) la courbure de Ricci est bornée supérieurement.
(ii) la courbure de Ricci est bornée inférieurement et le rayon d'injectivité est positif.
De plus, un soliton de Ricci contractant complet est de type topologique fini si sa coubure scalaire est bornée. ***Pour citer cet article : F.-q. Fang et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

In 1968, J. Milnor [8] conjectured that a complete non-compact Riemannian manifold with non-negative Ricci curvature has a finitely generated fundamental group. However, such a manifold may not have finite topological type. Examples of complete non-compact manifold with positive Ricci curvature without finite topological type was constructed by Gromoll–Meyer [5]. It has been an interesting topic in Riemannian geometry to study the topology of complete manifolds with positive (non-negative) Ricci curvature.

E-mail addresses: fuquan_fang@yahoo.com (F.-q. Fang), zhleigo@yahoo.com.cn (Z.-l. Zhang).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.03.021

In this Note we are concerned with complete Riemannian manifold (M, g) satisfying that $\mathrm{Ric} + \mathrm{Hess}(f) \geqslant \lambda g$ for some constant $\lambda > 0$ and $f \in C^\infty(M)$, i.e., whose Bakry–Émery Ricci tensor is bounded below by λ in the sense of [7]. When the equality holds, the manifold is a shrinking Ricci soliton, i.e., a self-similar solution of the well-known Ricci flow equation. If f is constant, Bakry–Émery Ricci tensor reduces to the Ricci tensor, and so the classical Myers' theorem implies that M is compact with finite fundamental group. In general, M may not be compact, but from the work of [1,3,4,6,7,10,9,11] etc., M still has finite fundamental group. Under some additional conditions, the manifold may be of finite topological type:

Theorem 1.1. *Suppose (M, g) is a complete Riemannian manifold satisfying $\mathrm{Ric} + \mathrm{Hess}(f) \geqslant \lambda g$ for some constant $\lambda > 0$ and $f \in C^\infty(M)$. Then M is of finite topological type, if either of the following alternative conditions holds*:

(i) $\mathrm{Ric} \leqslant Cg$ *for some constant* $C < \infty$;
(ii) $\mathrm{Ric} \geqslant -\delta^{-1} g$ *and the injectivity radius* $\mathrm{inj}(M, g) \geqslant \delta > 0$ *for some* $\delta > 0$.

If (M, g) is a shrinking Ricci soliton, then the Ricci curvature bounds can be relaxed by scalar curvature.

Theorem 1.2. *Suppose (M, g) is a complete shrinking Ricci soliton $\mathrm{Ric} + \mathrm{Hess}(f) = \frac{g}{2}$, where $f \in C^\infty(M)$. If the scalar curvature R is bounded, then M has finite topological type.*

In view of Theorem 1.2 it is natural to pose the following:

Conjecture 1.3. *Any shrinking Ricci soliton has finite topological type.*

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.

2. Proof of Theorem 1.1

Let (M, g) be such a manifold satisfying that $\mathrm{Ric} + \mathrm{Hess}(f) \geqslant \lambda g$ for some $\lambda > 0$ and $f \in C^\infty(M)$. By the deformation lemma of Morse theory, to prove Theorem 1.1, it suffices to show that the function f is proper and has no critical points outside of a compact set.

First fix one point $p \in M$ as a base point. For any $q \in M$ with $d(p, q) = L$, choose a shortest geodesic γ from p to q parametrized by arc length. Then

$$\langle \nabla f, \dot\gamma \rangle(q) = \langle \nabla f, \dot\gamma \rangle(p) + \int_0^L \frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\gamma(t)\big)\, dt \geqslant \langle \nabla f, \dot\gamma \rangle(p) + \int_0^L \big(\lambda - \mathrm{Ric}(\dot\gamma, \dot\gamma)\big)\, dt$$
$$\geqslant \lambda L - |\nabla f|(p) - \int_0^L \mathrm{Ric}(\dot\gamma, \dot\gamma)\, \mathrm{d}t.$$

If the integral $\int_0^L \mathrm{Ric}(\dot\gamma, \dot\gamma)\, \mathrm{d}t \leqslant \Lambda$ for some constant Λ independent of q and the choice of γ, then $|\nabla f|(q) \geqslant \langle \nabla f, \dot\gamma \rangle(q) \geqslant \lambda d(p, q) - |\nabla f|(p) - \Lambda$, which implies that $|\nabla f|(q)$ has a linear growth in $d(p, q)$ and so f is a proper function without critical points outside of a compact set. In the remainder of this section, we will focus on proving that $\int_0^L \mathrm{Ric}(\dot\gamma, \dot\gamma)$ has an upper bound under the assumptions of Theorem 1.1.

Case (i): $\mathrm{Ric} \leqslant Cg$ for some constant $C < \infty$. By Lemma 2.2 of [10], the integral bound is given by $\Lambda = 2(n-1) + 2C$.

Case (ii): $\mathrm{Ric} \geqslant -\delta^{-1} g$ and $\mathrm{inj}(M, g) \geqslant \delta > 0$ for some $\delta > 0$. Suppose $d(p, q) = L \geqslant \delta$. Let $\varphi(t) : [0, L] \to [0, 1]$ be an arcwise smooth function such that $\varphi(0) = \varphi(L) = 0$. By the second variation formula, as proposed in [10] or [11], we have the estimate: $\int_0^L \varphi^2(t)\, \mathrm{Ric}(\dot\gamma, \dot\gamma)\, \mathrm{d}t \leqslant (n-1) \int_0^L |\dot\varphi|^2\, \mathrm{d}t$. Now define φ by

$$\varphi(t) = \frac{3}{\delta} t, \quad t \in \left[0, \frac{\delta}{3}\right]; \quad \varphi(t) = 1, \quad t \in \left[\frac{\delta}{3}, L - \frac{\delta}{3}\right]; \quad \varphi(t) = \frac{3}{\delta}(L - t), \quad t \in \left[L - \frac{\delta}{3}, L\right],$$

then we have the estimate

$$\int_0^L \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t \leqslant (n-1)\int_0^L |\dot\varphi|^2\,\mathrm{d}t + \int_0^{\delta/3}(1-\varphi^2)\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t + \int_{L-\delta/3}^L (1-\varphi^2)\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t$$
$$\leqslant \frac{6}{\delta}(n-1)+\frac{2}{3}+\int_0^{\delta/3}\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t + \int_{L-\delta/3}^L \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t,$$

where in the second inequality, we used the fact that

$$-\int_0^{\delta/3}\varphi^2\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t \leqslant \frac{1}{\delta}\int_0^{\delta/3}\varphi^2\,\mathrm{d}t = \frac{1}{9},\quad \text{and similarly,}\quad -\int_{L-\delta/3}^L \varphi^2\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t \leqslant \frac{1}{9}.$$

We next prove that $\int_0^{\delta/3}\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t$ and $\int_{L-\delta/3}^L \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t$ are bounded from above and so finish the proof of Theorem 1.1. This is given by the following lemma:

Lemma 2.1. *If* $\mathrm{Ric} \geqslant -\delta^{-1}g$ *and* $\mathrm{inj}(M,g)\geqslant\delta>0$ *for some* $\delta>0$*, then*

$$\int_0^{\delta/3}\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t \leqslant \frac{6}{\delta}(n-1)+\frac{2}{3}$$

for any minimal arc length parametrized geodesic $\gamma:[0,\frac{\delta}{3}]\to M$.

Proof. Firstly, by $\mathrm{inj}(M,g)\geqslant\delta$, we can extend the geodesic γ to a shortest geodesic $\sigma:[0,\delta]\to M$, such that $\gamma(t)=\sigma(t+\frac{\delta}{3})$, $t\in[0,\frac{\delta}{3}]$. Set $L=\delta$ in the arguments above, we have $\int_0^\delta \varphi^2\,\mathrm{Ric}(\dot\sigma,\dot\sigma)\,\mathrm{d}t \leqslant (n-1)\int_0^\delta |\dot\varphi|^2\,\mathrm{d}t = \frac{6}{\delta}(n-1)$, then using $\mathrm{Ric}\geqslant-\delta^{-1}g$, we get the estimate

$$\int_0^{\delta/3}\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t = \int_{\delta/3}^{2\delta/3}\mathrm{Ric}(\dot\sigma,\dot\sigma)\,\mathrm{d}t \leqslant \frac{6}{\delta}(n-1) - \int_0^{\delta/3}\varphi^2\,\mathrm{Ric}(\dot\sigma,\dot\sigma)\,\mathrm{d}t - \int_{2\delta/3}^{\delta}\varphi^2\,\mathrm{Ric}(\dot\sigma,\dot\sigma)\,\mathrm{d}t$$
$$\leqslant \frac{6}{\delta}(n-1)+\frac{2}{3}. \qquad \square$$

3. Proof of Theorem 1.2

As before, we will prove that the potential function f to the Ricci soliton is proper and has no critical points outside of $B(p,\rho)$ for large ρ. Suppose (M,g) is a complete shrinking Ricci soliton which satisfies $\mathrm{Ric}+\mathrm{Hess}(f)=\frac{g}{2}$ for some potential function f. Suppose further that the scalar curvature $|R|\leqslant C$ for some constant $C<\infty$. It's well-known that the following analytic equality holds for the soliton (after modifying f by a translation, see [2] for example):

$$R+|\nabla f|^2 = f. \tag{1}$$

Let $q\in M$ be one point and denote by $\rho=d(p,q)$ the distance from p to q. Let γ be a shortest arc length parametrized geodesic from p to q. First we have:

Lemma 3.1. $\frac{\rho}{2}-|\nabla f|(p)-|\nabla f|(q)\leqslant\int_0^\rho \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t$.

Proof. It follows from a direct computation,

$$|\nabla f|(q)\geqslant\langle\nabla f,\dot\gamma\rangle(q)=\langle\nabla f,\dot\gamma\rangle(p)+\int_0^\rho \frac{\mathrm{d}^2}{\mathrm{d}t^2}f\big(\gamma(t)\big)\,\mathrm{d}t \geqslant -|\nabla f|(p)+\int_0^\rho\bigg(\frac{1}{2}-\mathrm{Ric}(\dot\gamma,\dot\gamma)\bigg)\,\mathrm{d}t. \qquad \square$$

On the other hand, by second variation formula as did in above section, we can get an upper bound of $\int_0^\rho \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t$. Precisely, for the function ψ defined by

$$\psi(t)=t,\quad t\in[0,1];\qquad \psi(t)\equiv 1,\quad t\in[1,\rho-1];\qquad \psi(t)=\rho-t,\quad t\in[\rho-1,\rho],$$

we have the estimate

$$\begin{aligned}\int_0^\rho \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t &\leqslant \int_0^\rho (n-1)|\dot\psi|^2\,\mathrm{d}t+\int_0^1(1-\psi^2)\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t+\int_{\rho-1}^\rho(1-\psi^2)\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t\\ &\leqslant 2(n-1)+\sup_{B(p,1)}|\mathrm{Ric}|+\int_{\rho-1}^\rho(1-\psi^2)\left(\frac12-\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(\gamma(t))\right)\mathrm{d}t.\end{aligned}$$

By integration by parts, we have the estimate for the last term

$$\int_{\rho-1}^\rho(1-\psi^2)\left(\frac12-\frac{\mathrm{d}^2}{\mathrm{d}t^2}f\big(\gamma(t)\big)\right)\mathrm{d}t\leqslant 1+2\int_{\rho-1}^\rho \psi\frac{\mathrm{d}}{\mathrm{d}t}f\big(\gamma(t)\big)\,\mathrm{d}t=1-2f\big(\gamma(\rho-1)\big)+2\int_{\rho-1}^\rho f\big(\gamma(t)\big)\,\mathrm{d}t.$$

Substituting this equality into above estimate, we obtain:

$$\begin{aligned}\int_0^\rho \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t &\leqslant 2n+\sup_{B(p,1)}|\mathrm{Ric}|+2\int_{\rho-1}^\rho f\big(\gamma(t)\big)\,\mathrm{d}t-2f\big(\gamma(\rho-1)\big)\\ &\leqslant 2n+\sup_{B(p,1)}|\mathrm{Ric}|+\sup_{x,y\in B(q,1)}2\big|f(x)-f(y)\big|.\end{aligned}\tag{2}$$

Now we are ready to show f is proper. By assumption $|R|\leqslant C$ and then using Eq. (1), we see that $|\nabla f|\leqslant\sqrt{f-R}\leqslant\sqrt{f+C}$. It follows that $|\sqrt{f(x)+C}-\sqrt{f(q)+C}|\leqslant\frac12$ for all $x\in B(q,1)$, and so

$$\begin{aligned}&\big|f(x)-f(y)\big|=\big|\sqrt{f(x)+C}+\sqrt{f(y)+C}\big|\cdot\big|\sqrt{f(x)+C}-\sqrt{f(y)+C}\big|\leqslant 2\sqrt{f(q)+C}+1,\\ &\quad\forall x,y\in B(q,1).\end{aligned}$$

The combination of Lemma 3.1, Eq. (2) and above estimate gives

$$\begin{aligned}\frac{\rho}{2}&\leqslant 2n+|\nabla f|(p)+|\nabla f|(q)+\sup_{B(p,1)}|\mathrm{Ric}|+4\sqrt{f(q)+C}+2\\ &\leqslant 2(n+1)+|\nabla f|(p)+\sup_{B(p,1)}|\mathrm{Ric}|+5\sqrt{f(q)+C}.\end{aligned}$$

It concludes that f is a proper function and then by boundedness of R and Eq. (1), f has no critical points outside of a compact domain. This finishes the proof of the theorem.

References

[1] W. Ambrose, A theorem of Myers, Duke Math. J. 24 (1957) 345–348.
[2] H.D. Cao, R. Hamilton, T. Ilmanen, Gauss densities and stability for some Ricci solitons, arXiv: math.DG/0404165.
[3] A. Derdzinski, A Myers-type theorem and compact Ricci solitons, Proc. Amer. Math. Soc. 134 (2006) 3645–3648.
[4] M. Fernández-López, E. García-Río, A remark on compact Ricci solitons, Math. Ann. 340 (2008) 893–896.
[5] D. Gromoll, W.T. Meyer, Examples of complete manifolds with positive Ricci curvature, J. Differential Geom. 21 (2) (1985) 195–211.
[6] X.M. Li, On extensions of Myers' theorem, Bull. London Math. Soc. 27 (1995) 392–396.
[7] J. Lott, Some geometric properties of the Bakry–Émery–Ricci tensor, Comment. Math. Helv. 78 (2003) 865–883.
[8] J. Milnor, A note on curvature and fundamental group, J. Differential Geometry 2 (1968) 1–7.
[9] G.F. Wei, W. Wylie, Comparison geometry for the Bakry–Émery Ricci tensor, arXiv: math.DG/0706.1120v1.
[10] W. Wylie, Complete shrinking Ricci solitons have finite fundamental group, Proc. Amer. Math. Soc. 136 (2008) 1803–1806.
[11] Z.L. Zhang, On the finiteness of the fundamental group of a compact shrinking Ricci soliton, Colloq. Math. 107 (2007) 297–299.