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Algebraic Geometry

Brauer obstruction for a universal vector bundle

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Abstract

Let X be a smooth complex projective curve with $\text{genus}(X) > 2$, and let $\mathcal{M}$ be the moduli space parametrizing isomorphism classes of stable vector bundles E over X of rank r with $\bigwedge^r E = \xi$, where ξ is a fixed line bundle. We prove that the Brauer group $\text{Br}(\mathcal{M})$ is $\mathbb{Z}/n\mathbb{Z}$, where $n = \text{g.c.d.}(r, \text{degree}(\xi))$. Moreover, $\text{Br}(\mathcal{M})$ is generated by the class of the projective bundle over $\mathcal{M}$ of relative dimension $r-1$ obtained by restricting the universal projective bundle over $X \times \mathcal{M}$ to a point of X. ***To cite this article: V. Balaji et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Obstruction de Brauer pour un fibré vectoriel universel. Soit X une courbe projective lisse de genre $g(X) > 2$ et soit $\mathcal{M}$ l'espace de modules paramétrant les fibrés vectoriels E stables sur X de rang r et ayant déterminant $\bigwedge^r E = \xi$, où ξ est un fibré en droites donné. Nous montrons que le groupe de Brauer $\text{Br}(\mathcal{M})$ est égale à $\mathbb{Z}/n\mathbb{Z}$, où $n = \text{pgcd}(r, \deg \xi)$. De plus $\text{Br}(\mathcal{M})$ est engendré par la classe du fibré projectif sur $\mathcal{M}$ de dimension relative $r-1$, obtenu par restriction du fibré projectif universel sur $X \times \mathcal{M}$ en un point de X. ***Pour citer cet article : V. Balaji et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Brauer groups of moduli spaces

Let X be a connected smooth projective curve defined over $\mathbb{C}$ such that $g(X) := \text{genus}(X) \geqslant 2$. Fix an integer $r \geqslant 2$; if $g(X) = 2$, then take $r \geqslant 3$. Fix an algebraic line bundle ξ over X of degree d. Let $\mathcal{M}(r, \xi)$ denote the moduli space parametrizing all isomorphism classes of stable vector bundles E over X of rank r with $\bigwedge^r E \cong \xi$. For notational convenience the variety $\mathcal{M}(r, \xi)$ will simply be denoted by $\mathcal{M}$. There is a natural universal projective bundle over $X \times \mathcal{M}$ of relative dimension $r-1$, which we will denote by $\mathbb{P}$. For any stable vector bundle $E \in \mathcal{M}$ and any point $x \in X$, the fiber of $\mathbb{P}$ over $x \times \{E\}$ is canonically identified with $P(E_x)$, the variety of one dimensional subspaces of E_x. For any closed point $x \in X$, the restriction of $\mathbb{P}$ to $\{x\} \times \mathcal{M}$ will be denoted by $\mathbb{P}_x$.

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Let $\mathrm{Br}(\mathcal{M})$ denote the *cohomological Brauer group* $H^2(\mathcal{M}_{et}, \mathbb{G}_m)$ of $\mathcal{M}$. Let $\beta \in \mathrm{Br}(\mathcal{M})$ be the class of the projective bundle $\mathbb{P}_x$. Recall that $\mathrm{Pic}(\mathcal{M}) = \mathrm{NS}(\mathcal{M}) = \mathbb{Z}$ by [4].

Remark 1.1. (i) The class β is the class of the gerbe which gives the obstruction to the existence of a universal vector bundle. For every étale morphism $e : U \to \mathcal{M}$ let $\mathcal{G}_U$ be the category of vector bundles E on $X \times U$ which are stable of rank r and determinant ξ on the fibers of $X \times U \to U$ such that $f_E = e$; the morphisms in $\mathcal{G}_U$ are the vector bundle isomorphisms. These categories are the fibers of a fibered category which has the structure of a $\mathbb{G}_m$ gerbe over the étale site of $\mathcal{M}$. We have an equivalence from $\mathcal{G}$ to the gerbe $b(\mathbb{P}_x)$ defined in [6, V4.8].

(ii) Under our conventions, the cohomology class associated to an Azumaya algebra of degree d on a scheme coincides with the class associated to the Brauer–Severi variety parametrizing rank d right ideals; this agrees with [6] and differs from [1].

Proposition 1.2. (a) *The group* $\mathrm{Br}(\mathcal{M})$ *is generated by* β*,* (b) *the variety* $\mathcal{M}$ *is simply connected, and* (c) *the natural homomorphism* $\mathrm{Pic}(\mathcal{M}) \to H^2(\mathcal{M}, \mathbb{Z})$ *is an isomorphism.*

Proof. By [4], there is an open subset $\mathbb{P}^s_\xi$ in a projective space together with a surjective morphism $f : \mathbb{P}^s_\xi \to \mathcal{M}$ satisfying the following condition: for any complex reduced variety Y and a vector bundle E on $X \times Y$, such that for all $y \in Y$, the restriction $E|_{X \times y}$ is a stable vector bundle of rank r with determinant ξ, the natural morphism $Y \to \mathcal{M}$ can be Zariski locally lifted to a morphism to $\mathbb{P}^s_\xi$. From [4, p. 89, Proposition 7.13] we have $\mathrm{Pic}(\mathbb{P}^s_\xi) = \mathbb{Z}$, so $\mathbb{P}^s_\xi$ is the complement of a Zariski closed subset of codimension $\geqslant 2$ in a projective space.

Since the projective bundle $\mathbb{P}_x \to \mathcal{M}$ pulled back to $\mathbb{P}_x$ is associated to a vector bundle, we see that there is a vector bundle $\mathcal{E}$ on $X \times \mathbb{P}_x$ such that for every $y \in \mathbb{P}_x$ the vector bundle $\mathcal{E}|_{X \times y}$ is stable of rank r and determinant ξ, and its isomorphism class corresponds to the image of y in $\mathcal{M}$. Hence there is a nonempty Zariski open subset U of $\mathbb{P}_x$ and a commutative diagram of maps

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{P}^s_\xi \\ \downarrow & & \downarrow \\ \mathbb{P}_x & \longrightarrow & \mathcal{M}. \end{array}$$

On the other hand, for any projective bundle $\widetilde{\mathcal{P}}$ over a connected regular scheme Z, by [5, p. 193], there is an exact sequence

$$\mathbb{Z} \cdot \mathrm{cl}(\widetilde{\mathcal{P}}) \longrightarrow \mathrm{Br}(Z) \longrightarrow \mathrm{Br}(\widetilde{\mathcal{P}}) \longrightarrow 0. \tag{1}$$

Hence the above commutative diagram, together with the facts that $\mathrm{Br}(\mathbb{P}^s_\xi) = 0$ (since $\mathbb{P}^s_\xi$ is the complement of a Zariski closed subset of codimension $\geqslant 2$ in a projective space) and the pullback homomorphism $\mathrm{Br}(\mathbb{P}_x) \to \mathrm{Br}(U)$ is injective, prove part (a).

Part (a) implies that $\mathrm{Br}(\mathcal{M})$ is a finite group. Hence part (c) follows using [7, p. 145, (8.7)].

Part (b) follows by applying π_1 to the above commutative diagram and observing that $\mathbb{P}^s_\xi$ is simply connected, the homomorphism $\pi_1(U) \to \pi_1(\mathbb{P}_x)$ is surjective and $\pi_1(\mathbb{P}_x) \simeq \pi_1(\mathcal{M})$. □

Remark 1.3. From Proposition 1.2(c) it follows that $H^2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. Using the finiteness of $\mathrm{Br}(\mathcal{M})$ together with the Kummer sequence and the comparison of classical and étale cohomology, it follows (cf. [7, p. 146, (8.9)]) that $\mathrm{Br}(\mathcal{M})$ is identified with the torsion subgroup $H^3(\mathcal{M}, \mathbb{Z})_{\mathrm{tor}}$ of $H^3(\mathcal{M}, \mathbb{Z})$.

Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Then there exists an integer m_0 such that for every integer $m \geqslant m_0$ and every semistable vector bundle V of rank r and degree d on X, the vector bundle $V(m) = V \otimes \mathcal{O}_X(m)$ is generated by its global sections, and furthermore, $h^1(V(m)) = 0$. Fix some $m \geqslant m_0$, and define $q = \dim H^0(X, V(m))$. Using the notation of [4], the universal bundle $\mathbb{F}$ on $X \times R^s$ has a $\mathrm{GL}(q)$ linearization and the projectivization $P(\mathbb{F})$ descends to $X \times \mathcal{M}$, and the descended projective bundle is identified with $\mathbb{P}$; see [4, pp. 61–62].

For a $\mathrm{GL}(q)$-linearized line bundle L on R^s, by $e(L)$ we will denote the integer that satisfies the following condition: the center $\mathbb{C}^* \subset \mathrm{GL}(q)$ acts on L_y, $y \in R^s$, by the character $t \mapsto t^{e(L)}$.

Proposition 1.4. *[4, p. 75, Proposition 5.1] Set* $n = \text{g.c.d.}(r, d)$*. Let* p *be an integer. Then there exists a* $\mathrm{GL}(q)$*-linearized line bundle* L *on* R^s *such that* $e(L) = p$ *if and only if* p *is a multiple of* n*.*

Lemma 1.5. *[1, p. 203, Proposition 4.4(ii)] For an everywhere nonzero vector bundle* V *on a scheme, consider the exterior power representation* $\mathrm{PGL}(V) \mapsto \mathrm{PGL}(\bigwedge^m(V))$*,* $0 \leqslant m \leqslant \mathrm{rank}(V)$*. Let* β_V *be the Brauer class of the projective bundle associated to a principal* $\mathrm{PGL}(V)$*-bundle for the standard action on* $P(V)$*. Then the Brauer class of the associated* $\mathrm{PGL}(\bigwedge^m V)$*-bundle is given by* $m \cdot \beta_V$*.*

Proposition 1.6. *If* β *is the Brauer class of* $\mathbb{P}_x$ *in* $\mathrm{Br}(\mathcal{M})$*, then* $n \cdot \beta = 0$*, where* n *is as in Proposition* 1.4.

Proof. We will denote by $\bigwedge^n \mathbb{P}$ the projective bundle over $X \times \mathcal{M}$ associated to the $\mathrm{PGL}(r, \mathbb{C})$-bundle $\mathbb{P}$ for the natural action of $\mathrm{PGL}(r, \mathbb{C})$ on $P(\bigwedge^n \mathbb{C}^r)$. The restriction of $\bigwedge^n \mathbb{P}$ to $\{x\} \times \mathcal{M}$ will be denoted by $\bigwedge^n \mathbb{P}_x$.

In view of Lemma 1.5, we need to show that $\bigwedge^n \mathbb{P}_x$ is Zariski locally trivial on $\mathcal{M}$. It is enough to show that $\bigwedge^n \mathbb{P}$ is Zariski locally trivial on $X \times \mathcal{M}$. This is equivalent to showing that the equivariant vector bundle $\bigwedge^n \mathbb{F}$ on $X \times R^s$ descends to a vector bundle after being tensored by a suitable equivariant line bundle. Now by Proposition 1.4, there exists an equivariant line bundle L on R^s such that $e(L) = n$. Further, $\mathbb{C}^*$ acts on $\bigwedge^n \mathbb{F}$ by $t \mapsto t^n$. Hence, the vector bundle $(\bigwedge^n \mathbb{F}) \otimes (p_{R^s})^*(L^*)$ has a trivial action of $\mathbb{C}^* \hookrightarrow \mathrm{GL}(q)$, and consequently it descends to $X \times \mathcal{M}$. This implies that $n \cdot \beta = 0$. □

Proposition 1.7. *Let* $0 < m < n$*. Then* $m \cdot \beta \neq 0$*.*

Proof. Suppose that $m \cdot \beta = 0$. We will get a contradiction. By Lemma 1.5, if $m \cdot \beta = 0$, it follows that $\bigwedge^m \mathbb{P}_x$ is the projectivization of some vector bundle V on $x \times \mathcal{M} = \mathcal{M}$. This implies the equivariant vector bundle $\bigwedge^m \mathbb{F}_x$ on R^s must be the pullback of V tensored with an equivariant line bundle on R^s. Hence there is an equivariant line bundle L on R^s with $e(L) = m$. Since $0 < m < n$ this is a contradiction (see Proposition 1.4). □

Propositions 1.2, 1.6 and 1.7 together give the following theorem:

Theorem 1.8. *The Brauer group* $\mathrm{Br}(\mathcal{M}) = \mathbb{Z}/n\mathbb{Z}$*. The Brauer group is generated by the Brauer class of* $\mathbb{P}_x$*.*

The above theorem remains valid for $g(X) = r = 2$ using the explicit descriptions of $\mathcal{M}$ in these cases [8].

2. Brauer group and stability

Let $\overline{M}$ be an irreducible normal complex projective variety of positive dimension. Fix a very ample line bundle ζ on $\overline{M}$. A nonempty Zariski open subset U of $\overline{M}$ will be called *big* if the complement $\overline{M} \setminus U$ is of codimension at least two. The smooth locus of $\overline{M}$, which is a big open subset, will be denoted by M. For any torsionfree coherent sheaf F defined on a big open subset $U \subset M$, the *degree* of F is defined to be the degree of F restricted to the general complete intersection curve obtained by intersecting hyperplanes on $\overline{M}$ from the complete linear system $|\zeta|$.

Let G be a complex reductive group. A principal G-bundle E_G defined over a big open subset $U \subset M$ is called *stable* if for all triples of the form (U', P, σ), where $U' \subset U$ is a big open subset of M, $P \subset G$ is a proper maximal parabolic subgroup, and $\sigma : U' \to E_G/P$ is a reduction of structure group of $E_G|_{U'}$ to P, the inequality $\mathrm{degree}(\sigma^* T_{\mathrm{rel}}) > 0$ holds, where T_{rel} is the relative tangent bundle for the natural projection $E_G/P \to U$; see [9,2,3]. A principal bundle defined over a big open subset is called a *rational principal bundle* (see [9,2]).

Now take $G = \mathrm{PGL}(n, \mathbb{C})$. Let $E_{\mathrm{PGL}(n,\mathbb{C})}$ be a principal $\mathrm{PGL}(n, \mathbb{C})$-bundle over a big open subset $U \subset M$. The projective bundle over U, of relative dimension $n - 1$, associated to $E_{\mathrm{PGL}(n,\mathbb{C})}$ for the natural action of $\mathrm{PGL}(n, \mathbb{C})$ on the projective space of lines in $\mathbb{C}^n$ will be denoted by E.

Lemma 2.1. *If the order of* E *in* $\mathrm{Br}(M)$ *is* n*, then the principal* $\mathrm{PGL}(n, \mathbb{C})$*-bundle* $E_{\mathrm{PGL}(n,\mathbb{C})}$ *is stable. In fact,* E *does not admit any reduction of structure group to any proper parabolic subgroup of* $\mathrm{PGL}(n, \mathbb{C})$ *over any big open subset of* M*.*

Proof. Any maximal parabolic subgroup of $\mathrm{PGL}(n,\mathbb{C})$ preserves a proper linear subspace of $\mathbb{CP}^{n-1}$. So a reduction of structure group of $E_{\mathrm{PGL}(n,\mathbb{C})}$ to a maximal parabolic subgroup is given by a linear subbundle of E. Let

$$\mathbb{L}_{U'} \subset E|_{U'} \tag{2}$$

be a linear subbundle over U' of relative dimension d, where $d \in [0, n-2]$.

Let E_d be the projective bundle over U associated to $E_{\mathrm{PGL}(n,\mathbb{C})}$ for the natural action of $\mathrm{PGL}(n,\mathbb{C})$ on the projective space $P(\bigwedge^{d+1}\mathbb{C}^n)$ of lines in $\bigwedge^{d+1}\mathbb{C}^n$. Using the natural embedding of the Grassmannian $\mathrm{Gr}(d+1,n)$ in $P(\bigwedge^{d+1}\mathbb{C}^n)$ we see that the Grassmann bundle over U parametrizing d dimensional linear subspaces in the fibers of the projective bundle E is embedded in E_d. Therefore, the subbundle $\mathbb{L}_{U'}$ in (2) gives a section of E_d over U'.

Note that $\mathrm{Br}(U') = \mathrm{Br}(M) = \mathrm{Br}(U)$ as U and U' are both big open subsets of M. Since the order of the class of E in $\mathrm{Br}(U')$ is n, and $d < n-1$, from Lemma 1.5 we know that the class of E_d in $\mathrm{Br}(U')$ is nonzero. This is in contradiction with the fact that we have a section of E_d over U'. Therefore, a subbundle as in (2) cannot exist. This completes the proof of the lemma. □

Recall that under the assumptions of Section 1, $\mathcal{M}$ is the smooth locus of a normal projective variety. From Theorem 1.8 and Lemma 2.1 it follows that the projective bundle $\mathbb{P}_x$ over $\mathcal{M}$ is stable provided the degree d is a multiple of the rank r.

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